Laboratory-Space and Configuration-Space Formulations of Quantum Mechanics, Versus Bell–Clauser–Horne–Shimony Local Realism, Versus Born's Ambiguity

- John F. Clauser

Abstract

A survey of more than thirty quantum mechanics and quantum field theory textbooks and review articles reveals two distinctly different schools of thought regarding what quantum mechanics is. Indeed, these books are found to promote two very different formulations of quantum mechanics. One is formulated in laboratory space, while the other is formulated in configuration space. Max Born (1933, 1935, 1969) appears to have been the founder and earliest promoter of the lab-space formulation. His textbook also acknowledges the two different possible formulations and pronounces them to be equivalent. He does so via what is herein called *Born's argument-space ambiguity*. His pronouncement is shown here to be false. The two formulations are, in fact, very different, and are herein shown to be incompatible. This article thus compares and contrasts these two formulations with each other, and with competing formulations from quantum field theory and with Bell-Clauser–Horne–Shimony Local Realism. Born's ambiguity is found to be embedded in quantum field theory also. Textbooks that promote the laboratory-space formulation are definitely easier to understand and visualize, in that they attempt to provide a "clean story line" that appears to explain many experiments. Born found the lab-space formulation preferable, because it is less mathematically abstract. Unfortunately, a lab-space formulation, a.k.a. a space–time formulation, is found to suffer from several serious fatal deficiencies. Paramount among these is that it can only describe single particle systems and cannot describe entanglement. Moreover, there does not appear to be any rigorous method to allow it to be extended to $N \geq 2$ particle systems, and thence to describe entanglement. Attempts in various books to do so are examined and found wanting. It is also noted that all charged particles in the world interact with each other, at least weakly, and are thus are always slightly entangled. Born's associated important construct, his "conserved probability current" is also examined. It is noted here that it may be constructed only in lab space, although various textbooks erroneously claim otherwise. Finally, it is noted that the lab-space formulation of quantum mechanics is really a form of Bell-Clauser–Horne–Shimony Local Realism, whereupon any successful attempt at a generalization to cover $N \geq 2$ particle systems will also violate experimental evidence that has been amassed against Local Realism. Alas, the conceptual model that lab-space formulations promote and use is found to be untenable. It is commonly, but incorrectly, suggested that a lab-space formulation is justified because it provides an approximation to a better theory, that is, to a formulation that is formulated in configuration space. That suggestion is unfounded. Unfortunately, the configuration-space formulation requires

a highly abstract mathematical formalism that is difficult to understand and that provides no conceptual model to allow its inner workings to be visualized.

Chapter 3
Laboratory-Space and Configuration-Space Formulations of Quantum Mechanics, Versus Bell–Clauser–Horne–Shimony Local Realism, Versus Born's Ambiguity

John F. Clauser

Introduction—What Quantum Mechanics is Not

A commonly asked question is "What is quantum mechanics?" Perhaps an easier question to answer is "What is Quantum Mechanics Not?" One answer to the first question that is commonly given is that "Quantum mechanics is a description of the world of the very small". Given observation of quantized magnetic flux, quantized vortices in liquid helium, and the Aharonov and Bohm effect, all over square-centimeter-sized areas, and especially given observation of two-particle quantum mechanical entanglement that has been observed to extend over distances of 133 and >1500 km, that answer also seems rather difficult to buy. Another answer that is commonly given is that "Quantum mechanics describes nature non-deterministically". Local Realism was first formulated by Clauser and Horne (1974), as an alternative theoretical framework to that of quantum mechanics. Like quantum mechanics, Local Realism includes theories that allow non-deterministic evolution. Quantum mechanics is thus hardly distinguished from Local Realism by that definition. Yet another answer is that quantum mechanics provides a description of nature in terms of "waves of probability (causally) propagating in space–time." That answer also will be shown here also to be incorrect.

A survey of more than thirty quantum mechanics and quantum field theory textbooks and review articles reveals two distinctly different schools of thought of what quantum mechanics is. These books are found to promote three very different formulations of quantum mechanics. One is quantum mechanics formulated in laboratory space, the second is quantum mechanics formulated in configuration space, and the third is quantum field theory, which is formulated in a mix of these of these spaces,

J. F. Clauser (✉)
817 Hawthorne Drive, Walnut Creek, CA 94596, USA
e-mail: john@jfcbat.com

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but differently by different authors. It is the purpose of this paper to describe, characterize, and contrast the configuration space and lab space formulations of quantum mechanics and quantum field theory with each other and with Local Realism.

The first of these schools is the laboratory-space school of quantum mechanics, or lab-space school, for short. In that school, Schrödinger's equation is formulated directly in lab space. Lab space is the ordinary three-dimensional space in which we live. The lab-space school's complex valued wave function $\Psi_{lab}(\mathbf{r}_{lab}, t)$ propagates causally as a wave in lab space. The real-valued probability density, $|\Psi_{lab}(\mathbf{r}_{lab}, t)|^2$, also propagates in lab space. It is described as propagating similarly to a classical field, except that it is subject to a somewhat mysterious "statistical interpretation", which was invented by Max Born. Via the lab-space school of thought, one is led to envision "waves of probability". That "interpretation" portrays $|\Psi_{lab}(r_{lab},t)|^2$ as a spatially dependent "probability density for finding the particle" at the point, $\mathbf{r}_{lab}$, that flows like a wavy fluid in space–time. Schiff (1955, p. 18) describes the waves and fluid as being similar to sound waves. Messiah (1961, p. 223) notes that "In the classical approximation, Ψ describes a fluid of non-interacting particles …, the density and current density of this fluid are at all times respectively equal to the probability density P and the probability current density $\mathbf{J}$ of the quantum particle at that point." Schiff (1955, p. 344) discusses dividing up all of (lab) space into cells and specifying the value of Ψ_{lab} at each cell. Schiff (1955, p. 348) says "*This application implies that we are treating* Eq. (6.16) [Schrödinger's equation in lab space] *as though it were a classical equation that describes the motion of some kind of material fluid.*"

Born asserts that this probability density may also be interpreted as the particle number-density of a beam of particles traveling through lab space. He famously showed that it may be used to demonstrate conservation of particle number in Rutherford scattering of the beam. Under the lab-space formulation the wave function's value (whatever it means) and the associated probability density's value are correspondingly defined at every space–time point, $(\mathbf{r}_{lab}, t)$, within the laboratory.

The quantum mechanics textbooks by Born (1933, 1935, 1969), Schiff (1955), Dicke and Wittke (1960), Merzbacher (1961, 1970), Eisberg (1961), Eisle (1964), Feynman and Hibbs (1965), Feynman's (1948) seminal review article, and French and Taylor (1978) all formulate quantum mechanics in lab space. Curiously, none of these books explicitly specifies or even calls notice to this important aspect of its description.[1] The wave-function's argument-space choice can be taken from context and from the supporting illustrations.[2] Typical wave and particle motions are depicted

[1] By exception Feynman and Hibbs (1965) say the book is based on a review article by Feynman (1948), titled "*A space–time approach to non-relativistic quantum mechanics.*" Schiff's (1955, p. 348) second quantization of $\Psi_{lab}(\mathbf{r}_{lab},t)$ admits that it treats the single-particle Schrödinger's equation "*as though it were a classical equation that describes the motion of some kind of material fluid*.".

[2] Authors that formulate quantum mechanics in configuration space almost always call the reader's attention to the argument space that is being used. On the other hand, authors that do not specify configuration space, instead generally use a lab space formulation. Merzbacher (1961, 1970) only introduces configuration space when he discusses quantum field theory (in his second edition), but

in Figs. 3.1, 3.2, 3.3, 3.4, 3.5, 3.6 and 3.7 that are reproduced from these books. French and Taylor (1978) further describe a large number of experiments whose results seem

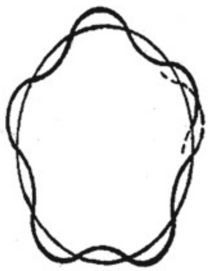

Fig. 18.—Motion of a wave in a circle; a single definite wave form is only possible when the circumference of the circle is a whole number of times the wave-length.

Fig. 3.1 Lab-space matter waves propagating around a nucleus, reproduced from Born (1933, 1935, 1969, Fig. 18, p. 132)

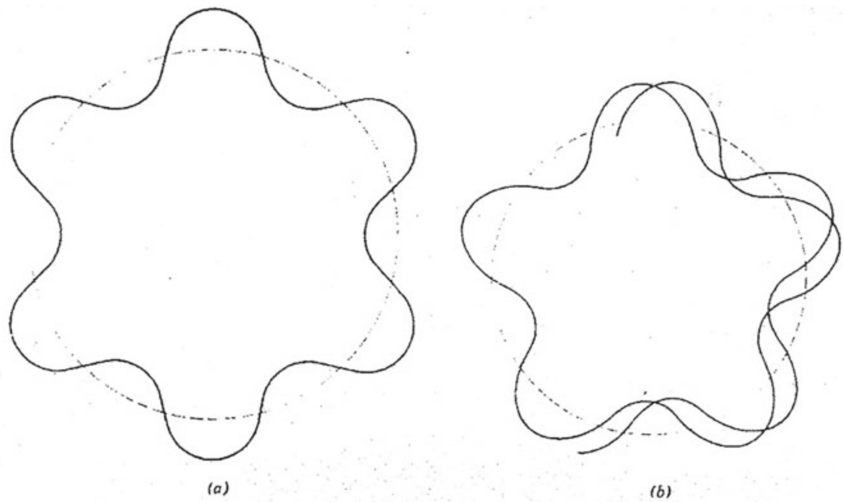

Figure 1.1. (*a*) Constructive interference of de Broglie waves in an atom distinguishes the allowed stable Bohr orbits. (*b*) Destructive interference of de Broglie waves in an atom disallows any orbit which fails to satisfy the quantum conditions.

Fig. 3.2 Lab-space matter waves propagating around a nucleus, reproduced from Merzbacher (1961, 1970, Fig. 1.1, p. 6). Eisberg (1961, 1967, Figs. 6–7, p. 152) also shows a very similar figure

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not until then (see also his quote in the section "Born's Argument-Space Ambiguity"). His vague lack of specificity is traced to Born's argument-space ambiguity, discussed in the section "Born's Argument-Space Ambiguity".

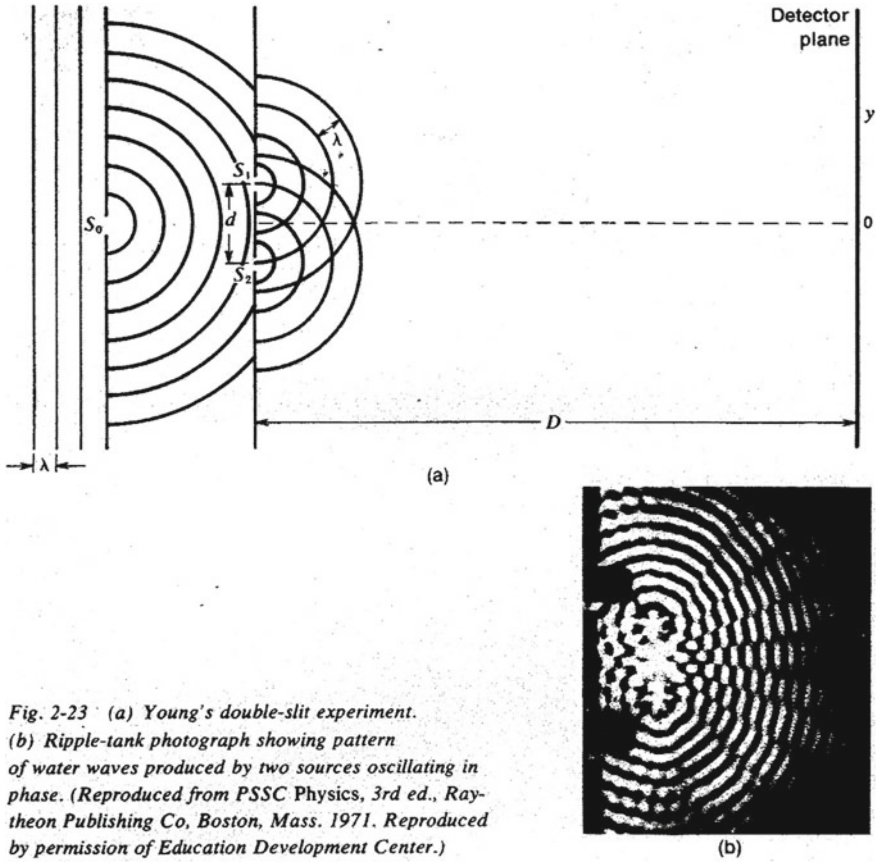

Fig. 3.3 Young's two-slit *Gedankenexperiment*, and photograph of wave motion in a ripple tank, reproduced from French and Taylor (1978, Fig. 2.3, p. 90)

to demand a lab-space explanation for the evident wave-like properties. By using lab space, the motion of fields and particles is readily visualized and understood. Abstract mathematics with no clear conceptual connection to the physics is generally avoided. Indeed, it was the stated intention by French and Taylor (see their preface) to use a description that presents a "clean story line". Schiff (1955) says, "we shall try to make the theoretical development seem plausible rather than unique."

Figures 3.1 and 3.2, are reproduced from Born (1933, 1935, 1969, Fig. 18, p. 132) and from Merzbacher (1961, 1970, Fig. 1.1, p. 6), respectively. They depict the motion of waves propagating in lab space around the nucleus of a hydrogen atom. Eisberg (1961, Fig. 6–7, p. 152) also offers a very similar figure, following his

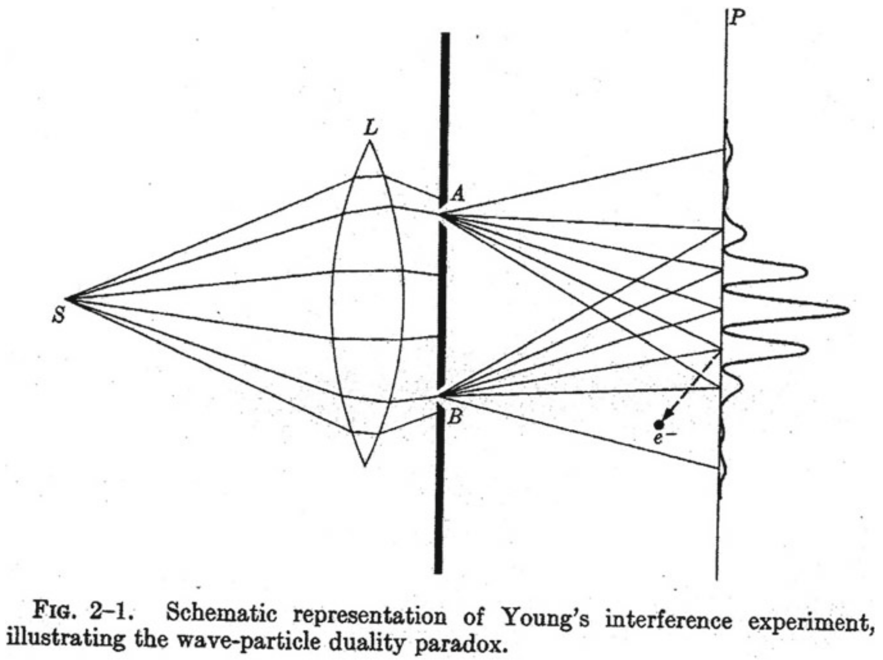

Fig. 3.4 Particles and waves propagating in lab space in Young's two-slit *Gedankenexperiment*, reproduced from Dicke and Wittke (1960, Fig. 2.1, p. 21)

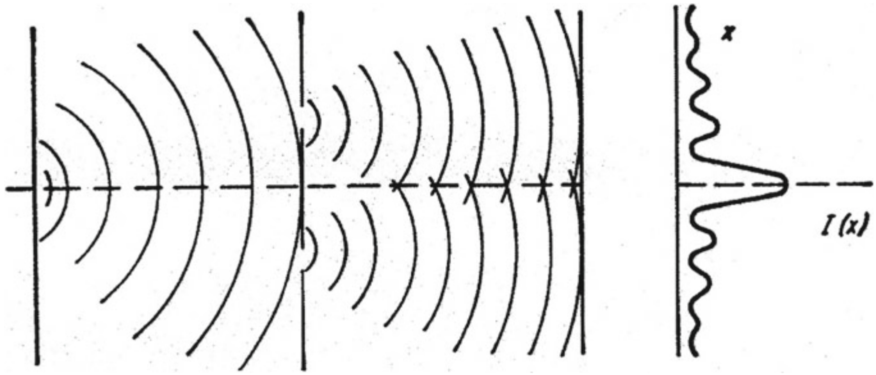

Fig. 3.5 Young's two-slit *Gedankenexperiment*, reproduced from Feynman and Hibbs (1965, Figs. 1-3, p. 5) showing propagation of Ψ waves in lab space

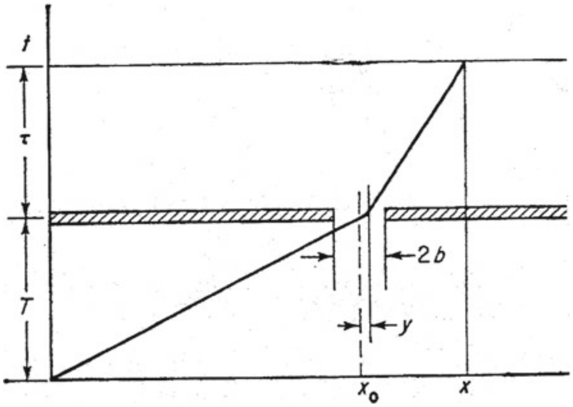

Fig. 3.6 Lab-space geometry used for calculating path integral solutions to the Helmholtz equation, reproduced from Feynman and Hibbs (1965, Figs. 3–3, p. 48)

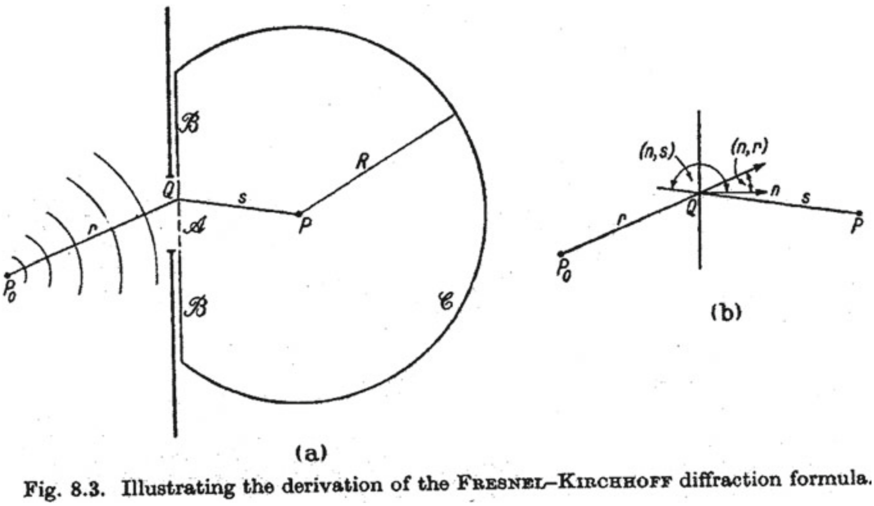

Fig. 3.7 Lab-space geometry used for calculating path integral summation solutions to the Helmholtz equation via the Fresnel-Kirchoff diffraction formula from classical physical-optics theory. It is reproduced from Born and Wolf (1959, 1987, Fig. 8.3), *Principles of optics*. It is for comparison with Fig. 3.6

promotion of deBroglie's pilot-waves lab-space formulation of quantum mechanics[3] (pp. 141–146). Figure 3.3 is reproduced from French and Taylor (1978, Fig. 2.3,

[3] deBroglie's pilot-waves formulation of quantum mechanics provided strong motivation for the development of Bell's Theorem and Local Realism. Unfortunately, it is soundly refuted by experimental evidence against Local Realism. See the section "Bell–Clauser–Horne–Shimony Local Realism".

p. 90). It depicts propagation in lab space of matter-waves through Young's two-slit *Gedankenexperiment*. It also compares this to a photograph of wave motion in a laboratory ripple tank. Figure 3.4, taken from Dicke and Wittke (1960, Fig. 2.1, p. 21), depicts motion of particles and waves propagating in lab space in Young's two-slit *Gedankenexperiment*. Figure 3.5, reproduced from Feynman and Hibbs (1965, Fig. 1–3, p. 5) depicts wave motion in lab space in Young's two-slit *Gedankenexperiment*. Figure 3.6, also taken from Feynman and Hibbs (1965, Fig. 3–3, p. 48), shows the lab-space geometry used by them for calculating path integral solutions to the time-independent Schrödinger's equation, which is the same partial differential equation as the Helmholtz equation. Figure 3.6 may then be directly compared with Fig. 3.7, which is reproduced from Born and Wolf's *Principles of Optics* (1959, 1987, Fig. 8.3). It shows the same lab-space geometry that they use for calculating path-integral solutions to the Helmholtz equation via the Fresnel-Kirchoff diffraction formula from classical physical-optics theory (in lab space and with no particles evident).

The lab-space formulation is reviewed below in the section "The Lab-Space Formulation of Quantum Mechanics". The formulation in lab space of the single particle Schrödinger's equation is described in the section "Single particle Schrödinger's equation in lab space". The section "Born's Probability Density and Conserved Probability Current Defined in Lab Space" describes the calculation of probabilities in lab space and Born's conserved probability current that also flows in lab space similarly to the flow of a fluid. This current was invented by Born to explain particle flux conservation in Rutherford scattering. It is only capable of being formulated in lab space, and is a concept that only makes sense in lab space.

The second school is the configuration-space school. It is described below in the section "The Configuration-Space Formulation of Quantum Mechanics". Schrödinger's equation is formulated using a highly abstract mathematical space. It describes a very general quantum mechanical system. That space is called configuration space. Its complex valued wave function $\Psi_{\text{config}}(q_{1,\text{config}}, \ldots, q_{k,\text{config}}, s_{1,\text{config}}, \ldots, s_{k,\text{config}}; t)$ is very different from $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}}, t)$. Rather than specifying the position within the lab where the wave function is to be evaluated, the wave-function's arguments instead specify the various degrees of freedom of the described system. For a system composed of N particles, the wave function, $\Psi_{\text{config,N}}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{\text{N,config}}, t)$, has arguments that specify then N positions, $\mathbf{r}_{1,\text{config}} - \mathbf{r}_{\text{N,config}}$, of these N particles. The arguments may be either continuously varying or discretely varying, and may include non-classical degrees of freedom, like spin, $s_{1,\text{config}}, \ldots, s_{k,\text{config}}$, and isotopic spin.[4] A discretely varying argument with M allowed values, in turn, may be used as an index, so that Ψ_{config} may be considered to be a vector-valued function with M components. Quantum mechanics textbooks that promote this school include the books by von Neumann (1932,1955), Landau and Lifshitz (1958, 1965), Messiah (1961) and Bjorken and Drell (1964). Interestingly, configuration-space-school textbook authors rarely acknowledge the existence of the lab-space formulation of quantum mechanics, and vice versa. Noteworthy exceptions are Dicke and

[4] See Bjorken and Drell (1964, p.222).

Wittke (1960) and Merzbacher (1961). Their comparisons of these formulations are described in the section "Born's Argument-Space Ambiguity".

There are also many configuration-space based quantum mechanics textbooks, which may be classified as applied quantum mechanics textbooks.[5] These books primarily discuss N-particle systems, whereupon this later feature forces them to use configuration space. Such books include the works by Pauling and Wilson (1935), Bethe and Salpeter (1957), and Condon and Shortley (1964). This latter group does not attempt to "explain" the N-particle Schrödinger's equation. Instead, the formalism is simply accepted as given and useful in calculating results that match experiment.

The configuration-space school is described in the section "The Configuration-Space Formulation of Quantum Mechanics". There are many additional configuration-space schools of thought to be found in various quantum mechanics books than are treated here in section "The Configuration-Space Formulation of Quantum Mechanics". Von Neumann's Chap. 1 formulation, listed therein as "*The Original Formulations*", is followed in the section "The Configuration-Space Formulation of Quantum Mechanics", along with that by Messiah (1961). Other configuration-space formulations that are not discussed, for example, include Heisenberg's matrix mechanics, outlined very nicely, for example, by Condon and Shortley (1964), Dirac's (1930, 1935, 1947) formulation of quantum mechanics in Hilbert space, and others. Each formulation is proclaimed by its promoters to be mathematically consistent with all of the others, and each is a progressively more mathematically abstract. Correspondingly, each is progressively more difficult to visualize than the others. A discussion of all of these other related formulations goes beyond the scope of the present article.

Configuration space is defined in the section "Configuration Space". The section "Schrödinger's Equation in Configuration Space" describes the configuration-space N-particle and single-particle ($N = 1$) Schrödinger's equations. Calculation of probabilities using the configuration-space formulation is described in the section "Calculation of Probabilities Using the Configuration-Space Formulation". Finally, the possible factorization of the N-particle configuration-space wave function that occurs when the various particles are strictly non-interacting and statistically independent is discussed in the section "Factorization of Schrödinger's N-particle Configuration-Space Wave Function". Particles that do not interact and that have never interacted with each other, in that special case, may be considered statistically independent. It is noted in the section "Wave-Function Factorization or Not!", however, that in the more general case, particles are usually entangled, even when there is (presently) no interaction between them. In general, they are not independent. It is also noted that all charged particles in the world interact with each other, at least weakly, and are thus are always slightly entangled, with that entanglement exponentially growing in time, so that the special case never accurately applies.

[5] This appellation is offered by Bethe and Salpeter (1957).

The section "Born's Argument-Space Ambiguity" proceeds to examine some of the various discrepancies between the lab-space and configuration-space formulations of quantum theory. For example, one may note that the depictions in Figs. 3.1–3.7 are of quantum mechanical matter-wave propagation and particle propagation in lab space, i.e. propagation in the three-dimensional space in which we live. Unfortunately, the diagrams in Fig. 3.1 (from Born) and Fig. 3.2 (from Merzbacher), depicting the motion of waves in lab space around the nucleus of a single-electron hydrogen atom, are impossible to be drawn for a two-electron helium atom! (See the section "Born's Ambiguity's Misuse by the Lab-Space Formulation School" for observations and discussions of this fact by Dicke and Wittke (1960), Merzbacher (1961, 1970). Woops! Lab space doesn't get very far up the periodic table in describing the structure of atoms, does it? That is because a lab-space wave function, $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}})$, is limited to describing only a single-particle system. Also, the lack of any evident spatial dependence by a two-particle configuration-space wave function, $\Psi_{\text{config},2}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}})$, or by its associated probability density $|\Psi_{\text{config},2}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}})|^2$, prohibits these quantities from being considered as a valid description of a wave-like field propagating in lab space for $N > 1$ electron atoms, as per Figs. 3.1 and 3.2, whereupon $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}})$ does not and cannot describe waves propagating in lab space for helium.

The lack of ability for a lab-space wave function to describe $N > 1$ particle systems becomes a fatal difficulty for the lab-space formulation, especially when entanglement is required. Curiously, the importance of this observation seems heretofore to have gone unnoticed. It is shown in the section "Born's Argument-Space Ambiguity" that it apparently stems from a somewhat hidden ambiguity introduced by Born. In his textbook, he wrongfully pronounces the equivalence of configuration-space and lab-space wave functions. It is shown in the section "Born's Ambiguity's Misuse by the Lab-Space Formulation School" that there appears to be no rigorous method to allow a lab-space wave function and the associated lab-space formulation of quantum mechanics to be extended to describe $N \geq 2$ particle systems. Attempts by various books to demonstrate a direct connection between $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}})$ and $\Psi_{\text{config},2}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}})$ are examined in the section "Born's Ambiguity's Misuse by the Lab-Space Formulation School" and are found wanting. Thus, the wave functions are not equivalent, despite Born's pronouncement. This lack of equivalence is herein called *Born's argument-space ambiguity.* Correspondingly, there is no rigorous method evident to allow a lab-space wave function, along with the lab-space formulation also to describe entanglement! Born's important construct, the "conserved probability current" is examined in the section "Born's Conserved Probability Current as Re-Interpreted Using the Configuration-Space Formulation", wherein it is shown that it may be constructed only in lab space, and only makes sense in lab space. It thus may not be used to describe a pair of particles in an entangled state, e.g. as a pair of coupled currents.

Born's ambiguity is produced by a sneaky slight-of-hand. The ambiguity manifests itself by using the same ambiguous (multiply defined) symbol $\mathbf{r}$ to represent two very different quantities, $\mathbf{r}_{\text{lab}}$ and $\mathbf{r}_{1,\text{config}}$, with different meanings altogether of their arguments. Then, two different equations (Schrödinger's equation for $N = 1$ particle in lab space and in configuration space) that are formally the same in their appearance are produced and both are claimed to govern nature. Both equations use the common ambiguous symbol **r**. Presto, since the equations formally appear to be the same, the equations are claimed to be equivalent, when in reality, they are not. The switch is done so seamlessly that no one is aware of the prestidigitation that has passed.

Details of the switch are outlined below in the section "Born's Ambiguity's Misuse by the Lab-Space Formulation School". Born is not alone in its use. It will then be seen in the section "Born's Argument-Space Ambiguity" that many of the quoted authors, in addition to Born, appear to have been ambiguous in their choice of propagation space. Messiah (1961) usually reminds the author when configuration space is being used, (but not always). He fails to do so in his discussions of scattering and quantum field theory. In his Chap. 6, *Classical Approximation and the WKB method*, he follows exactly this prescribed prestidigitation in his §4, *Classical Limit of the Schrödinger Equation* (pp. 222–228).

Born's notational ambiguity is revealed (and avoided) in the present article by simply and "inelegantly" displaying the different meanings of the symbols as they are used. The equations introduced in the sections "The Lab-Space Formulation of Quantum Mechanics" and "The Configuration-Space Formulation of Quantum Mechanics" are all taken verbatim from the above-mentioned quantum mechanics textbooks. However, they differ slightly and conspicuously in notation from those in the textbooks by the addition here of "Lab" and "config" subscripts. These obsequiously conspicuous "misquotations" are meant as a necessary clarification of these quotations. The subscripts are unceremoniously added to all symbols that describe a spatial and/or spatially dependent variable or operator. (Sorry for the necessary annoyance and loss of "elegance".) The additions extend to important symbols that are quoted from the various textbooks to indicate which space it is assumed that the quoted author is using, given that there are usually at least two possibilities to choose.

In the present article, in order to highlight Born's argument-space ambiguity, the subscript "config" is added to any dynamical variable or operator defined in a general configuration space, and the subscript "j, config" is added to a dynamical variable or operator defined for the j-th particle in an N-particle configuration space. Sometimes, where there still may be confusion (as in the section "Wave-Function Factorization or Not!"), the subscripts "particle1", "particle2", etc. are used to specify the setting of a specific particle's number index to a dummy index's value. Dependent variables that depend upon such dynamical variables, and/or upon operators in an N particle system, are given the subscript "config, N". This notation is used to prevent confusion of these variables with formally similar variables that are defined in lab space.

The sections "Quantum Field Theory 1—Quantization of Known Classical Fields" through "Quantum Field Theory 2—Second Quantization of Wave-Functions" examine quantum field theory. Standard quantum mechanics (via Born's conserved probability current) requires that the number of particles described by it to be constant. In nature, however, particles are created and annihilated by various processes, and these processes are not part of standard quantum mechanics. Standard quantum mechanics thus needs to be extended to account for a varying number of particles and to calculate how that number changes with time. Additionally, Einstein (1917) demonstrated an important need for what he called "*directional radiation bundles.*" These must exist as a new quantum mechanical component part of the electromagnetic field in order for the second law of thermodynamics to hold. These particle-like *directional radiation bundles,* now known as photons, are not present in the classical description of the electromagnetic field in terms of Maxwell's equations. A quantum mechanical modification thus needs to be added to the classical description of the field. Quantum field theory was correspondingly developed to handle these different evident requirements associated with photons, and with the creation and annihilation of massive particles. The first of these two different modifications to standard quantum mechanics is called Quantum Field Theory 1". It describes the quantization of known classical fields (light and sound) in the sections "Quantum Field Theory 1—Quantization of Known Classical Fields" through "Some Observations Regarding Which School Is "Proper"". The second, described in the section "Quantum Field Theory 2—Second Quantization of Wave-Functions", is called Quantum Field Theory 2. It describes the "second quantization" of matter-wave fields.

Fermi's (1932) formulation of the quantum theory of radiation is discussed in the section "Quantum Theory of Radiation and Quantum Electrodynamics". The relation of its quantized electromagnetic field to Einstein's (1917) need for *directional radiation bundles* is discussed in the section "Quantum Field Theory 1—Quantization of Known Classical Fields". It is noteworthy that Fermi (1932) tried to show that his quantized electromagnetic field demonstrated a causal behavior in real space–time. The section "von Neumann's Collapse of the Entangled Two-Photon Quantized Electromagnetic Field", however, shows that this aim cannot be achieved, because of the non-causal non-unitary evolution that it must undergo as required by von Neumann's collapse process. Said non-causal behavior is to be expected, and is predicted by experimental tests of Local Realism.

A scrutiny of Fermi's treatment reveals the existence of fields of two types. These two types are sometimes confused with each other via Born's ambiguity by others, but not by Fermi. Indeed, they are the same two types identified in standard quantum mechanics, i.e. they differ by their choice of argument space—lab space or configuration space. Things get worse. In Quantum Field Theory 2, various authors attempt to second quantize at least seven kinds of fields. It is thus observed in the section "Quantum Field Theory 2—Second Quantization of Wave-Functions" that second quantization of the matter-wave field evidently cannot proceed without a liberal use of Born's ambiguity.

Inspired by Bell's (1964) paper[6] and following the associated proposed experimental testing of local hidden variable theories by Clauser et al. (1969), Clauser and Horne (1974), added yet a fourth candidate school of thought for describing natural phenomena. Clauser and Horne (CH) originally named their formulation "Objective Local Theories". Clauser and Shimony (1978) reviewed it and renamed it "Local Realism".

[6] Bell (1964), in turn, was inspired by his reanalysis of Einstein, Podolsky, and Rosen (1935).

Bell–Clauser–Horne–Shimony Local Realism is formulated in lab space and provides experimental predictions that differ from those made by all of the above schools of quantum mechanics. Importantly, that feature allows Local Realism to be distinguished experimentally from those three schools of thought.

Local Realism's most attractive heuristic feature is that it provides very general lab space formulation for all theories of natural phenomena that attempt to describe real stuff in a real space–time framework, consistently with special relativity. It's most disappointing but also important heuristic feature is that it is soundly refuted by experiment. Local Realism describes tangible stuff, stuff that is present locally in space–time, stuff that can be put in a box, and stuff that can be used for storing bits of information. It thus provides a general framework for the space–time description of this stuff. An important requirement for the description is that it does not allow communication among any of the stuff to occur at super-luminal velocities, so as to somehow influence the results of experiments. Thus, Local Realism is consistent with special relativity. As an extension of classical field theories, Local Realism further allows non-deterministic evolution of the stuff that it describes, and has no required or seemingly artificial or presumed limitations to the precision of measuring devices.

Local Realism carefully defines a class of theories that attempt to describe "*real stuff in real space–time*". By providing this definition along with its experimental predictions, Local Realism provides heuristic value in showing what quantum mechanics is not! Importantly, Local Realism provides experimental predictions that must be obeyed by any theory that attempts to describe "*real stuff in real space–tim*e". Also importantly, the Clauser–Horne inequality's predictions differ from those made by quantum mechanics. Starting in 1972 with the first experiment by Freedman and Clauser (1972), followed by the second one by Clauser (1976), and then followed by a long list of confirming experimental refinements, Local Realism has been soundly refuted by experiment. Clauser (2017) reviews a partial list of twenty experiments that have been performed at many different laboratories around the world during the period 1972 through 2013 to test Local Realism. All but one of these experiments refute Local Realism's predictions. The theories basic predictions along with acid tests of its prohibition of internal super-luminal communication have all been now tested experimentally. Thus, quantum mechanics does not describe *real stuff in real space–time.*

Curiously, the lab space formulation of quantum mechanics bears uncanny and disturbing similarities to Local Realism. The difficulties experienced by the lab-space formulation are thus to be expected, given that the lab space formulation attempts to describe Local Realism's *real stuff in real space–time.* Indeed, the lab space formulation of quantum mechanics actually qualifies as a theory of Local Realism. It slickly avoids experimental refutation similarly to Local Realism's refutation, because it is limited to describing only single particle systems. Such systems, in turn, cannot exhibit non-local entanglement. Correspondingly, the lab space formulation of quantum mechanics is then incapable of making experimental predictions that can be tested by these experiments. Nonetheless, that deficiency should hardly be considered a salvation of the lab-space quantum mechanics formulation's viability.

The article's conclusions are presented in the final section.

Laboratory Space and Classical Fields

In this section we define what we refer to as "lab space" and "classical fields". While these definitions may seem obvious, tedious, and perhaps even boring, it is important to clarify them before proceeding, since they are frequently blurred[7] by practitioners of quantum theory and mathematical physics.

Laboratory space or lab space, for short, is the three-dimensional space in which we live, and the space in which Euclidean geometry is understood. It is the space used by geometrical vectors and by classical vector and scalar fields. Every point within one's laboratory has a unique position in what is called lab space. Said point is depicted by the symbol $\mathbf{r}_{\mathrm{Lab}}$.

Geometric Vectors Are Defined in Lab Space

A classical geometrical vector (within in a classical vector field) is typically depicted as an arrow. It is a quantity with both a magnitude and a direction, and is commonly used to represent force and velocity. It is sometimes referred to as a "*vector of physics*". Its definition does not require the existence of a coordinate system, and physical laws are often expressed in vector notation without reference to a coordinate system (see Margenau and Murphy (1943, 1956, p. 139)). It is often quantified by specifying its real scalar Cartesian components as projections on some set of spatial axes. When a classical geometrical vector is specified by its components, doing so thus requires a simultaneously defined coordinate system that has three real-valued components (exactly equal in number to the number of dimensions of lab space). In a classical vector field, a geometrical vector or "vector of physics" is anchored at every point in lab space, its three components are all real valued.[8]

Given a unique arbitrarily chosen point in the lab that is called the origin, every point in the lab may be located relative to the origin via the use of an appropriate Cartesian coordinate system. Each point in the lab also may be located by using a "position" vector, $\mathbf{r}_{\mathrm{Lab}}$, that extends from the origin to it. Given a set of lab coordinates, every point in the laboratory thus has an associated vector position specified by

$$\mathbf{r}_{\mathrm{Lab}} = \hat{\mathbf{e}}_{\mathrm{x}} \mathrm{x}_{\mathrm{Lab}} + \hat{\mathbf{e}}_{\mathrm{y}} \mathrm{y}_{\mathrm{Lab}} + \hat{\mathbf{e}}_{\mathrm{z}} \mathrm{z}_{\mathrm{Lab}}, \tag{3.1}$$

where $\hat{\mathbf{e}}_{\mathrm{x}}$, $\hat{\mathbf{e}}_{\mathrm{y}}$, and $\hat{\mathbf{e}}_{\mathrm{z}}$ are three orthogonal Cartesian unit basis vectors extending from the origin. Note that lab space necessarily has three, <u>and only three</u>, spatial dimensions. As an alternative to a Cartesian coordinate system, a variety of other 3D

[7] Said blurring is commonly called "interpreting".

[8] The term "real" here means having no imaginary component.

coordinate systems may be defined within said space without modifying in any way the geometrical properties of the space, itself.

Classical Fields Are Defined in Lab Space

A classical field is typically represented as a mathematical function, $f(\mathbf{r}_{Lab})$, of position, $\mathbf{r}_{Lab}$. The function assigns a unique value, f, (or values) to every point in the lab, i.e. the function's dependent variable, f, specifies the field's value at the point, $\mathbf{r}_{Lab}$. Many different fields may exist simultaneously. The value(s), f, also may take many forms—scalar, vector, tensor, set of scalars, set of vectors, or even more general mathematical forms. That is, f may take any basic generalized-spaghetti form that might be needed to specify the field's property or properties. If needed, it may be an n-tuple of numbers that are all defined simultaneously at the point, $\mathbf{r}_{Lab}$. The function's independent variable, or argument, is here represented by the dummy variable, $\mathbf{r}_{Lab}$. It specifies the position in the lab where the function's value or values apply Classical fields may be time varying, whereupon a second argument, t, is used to specify the time at which the function's assignment applies. In general, lab space is required for the description of the mechanics of continua, such as fluids, when no particles are evident or even present.

Classical Fields Are Used to Specify How Classical Stuff is Distributed Throughout Lab Space

The assumption that nature consists of *real stuff* distributed throughout lab space is the fundamental basis of both classical physics and Local Realism. Classical fields are used by classical physics and Local Realism to describe how the properties of *real stuff* are distributed throughout lab space. So-called "classical physics" (definitions vary) then provides a theory (usually, but not necessarily deterministic) that is a subset of Local Realism. Local Realism will be described in more detail in section "Bell–Clauser–Horne–Shimony Local Realism". It allows for a non-deterministic evolution of stuff. As a preface to that section, it should be noted that all of the expressions used by Local Realism qualify as "classical fields" and are functions of $\mathbf{r}_{Lab}$. The real-valued probability density, $|\Psi_{lab}(\mathbf{r}_{lab}, t)|^2$, also qualifies as a classical field defined in lab space.

Familiar examples of classical vector fields include the velocity field of a fluid, $\mathbf{v}(\mathbf{r}_{Lab}, t)$, and the electric field $\mathbf{E}_{Lab}(\mathbf{r}_{Lab}, t)$ from electromagnetic theory. The velocity field's vector may be decomposed with its associated Cartesian components defined as

$$\mathbf{v}_{Lab} = \hat{\mathbf{e}}_x v_{x,Lab} + \hat{\mathbf{e}}_y v_{y,Lab} + \hat{\mathbf{e}}_z v_{z,Lab}. \tag{3.2}$$

Given some quantity of the stuff that is described by a field whose value varies as a function of position within the lab, there is an important differential vector operator that operates on the quantity's value (only) in lab space. It is the lab-space gradient operator:

$$\nabla_{\mathrm{Lab}} = \hat{\mathbf{e}}_{\mathrm{x}} \partial/\partial \mathrm{x}_{\mathrm{Lab}} + \hat{\mathbf{e}}_{\mathrm{y}} \partial/\partial \mathrm{y}_{\mathrm{Lab}} + \hat{\mathbf{e}}_{\mathrm{z}} \partial/\partial \mathrm{z}_{\mathrm{Lab}}. \tag{3.3}$$

It may be used to calculate the rate of change of a function's value with respect to spatial position within the lab. Important theorems of vector analysis (Gauss's theorem, Green's theorem(s), and Stokes theorem, etc.) all apply rigorously to scalar and vector fields that are defined in lab space. Waves in linear classical fields propagate in lab space, typically governed by the classical wave equation. In such cases, the time dependence of the equation can be factored out, and the result is the time-independent linear Helmholtz equation.

Lab Space is not the Same as a General Vector Space

At this point in the discussion, it is worth commenting on what lab space is not. Unfortunately, some authors have imprecisely adopted the use of the words "vector space" and "vector" to describe abstract mathematical constructs in quantum mechanics that may or may not include "lab space" and "classical geometrical vector" in lab space.

A "geometrical vector" is sometimes confused with an "algebraic vector". The two types of vectors are sometimes treated as equivalent interchangeable entities. They are not! Apostol (1961, V1, p. 252) discusses "*another approach to vector algebra called the abstract or axiomatic approach, …. Instead, vectors and vector operations are thought of as undefined concepts of which we know nothing except that they satisfy a certain set of axioms. Such an algebraic system, with appropriate axioms is called a linear space or a linear vector space. …*" An algebraic vector (within a vector space) is commonly represented as an "*ordered n-tuple*" of numbers. The number, n, of numbers in the n-tuple may be greater than three, even infinite, and the numbers themselves may be complex.

An ordered n-tuple of numbers is not, in general, associated with a specific position in lab space. It does not necessarily qualify as a geometrical vector, especially when n is greater than three, and especially when the numbers themselves are complex. For the purposes of the present discussion, a "vector space" is taken here to mean a set of numbers (an algebraic vector) that somehow functionally depends on some other set of numbers, and nothing more.

The lab-space gradient operator defined by (3.3), when operating on a lab-space classical scalar field produces a lab-space vector field with a unique geometric vector thereby defined at every point in lab space. On the other hand, an algebraic vector defined in configuration space, (in the section "The Configuration-Space Formulation of Quantum Mechanics") is not a lab-space geometric vector. As a preface to that section, it should be noted that the real-valued probability "density",

$|\Psi_{\text{config},N}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{N,\text{config}}, t)|^2$ correspondingly does not qualify as a classical field. Indeed, it is also pointed out in that section that Messiah (1961, p. 164) succinctly defines "*The wave functions of wave mechanics are the square integrable functions of configuration space.*" It should also be stressed that the special case with N = 1 does not change the space on which $\Psi_{\text{config},N}$ depends from configuration to lab.

Bell–Clauser–Horne–Shimony Local Realism

The term "Local Realism" describes a large class of theories, first explicitly defined by Clauser and Horne (CH) in 1974. Clauser and Horne named the class, "*Objective Local Theories*". Significant contributions were made to the theory by Bell and Shimony. In response to Clauser and Horne (1974), Bell offered a different (unpublished) version which he called "*The Theory of Local Beables*". Local Realism was refined and clarified jointly by Bell et al. (1976–1977), regarding methods used to specify the apparatus parameters and differences between "*Objective Local Theories*" and "*The Theory of Local Beables*". Local Realism theory was further refined in the review article by Clauser and Shimony (1978), who gave the class of theories the name Local Realism.

The Locality principle is based on special relativity. It asserts that nature does not allow the propagation of information faster than light to thereby influence the results of experiments. Without Locality, one must contend with paradoxical causal loops, as are now popular in science fiction thrillers involving time travel. Upholding Locality is effectively denying the possible reality of causal loops.

Realism is a philosophical view, according to which external reality is assumed to exist and have definite properties, whether or not they are observed by someone. Another way of describing what is meant by Realism is to say that it specifies that nature consists of "objects", i.e. of stuff that is distributed throughout space–time with "objective reality". Realism assumes that stuff, i.e. objects, with spatial positions and structure, exist and have inherent properties on their own. It does not require that these properties fully determine the results of an experiment locally performed on said stuff. Instead, in a possibly non-deterministic world, it simply allows the properties of stuff (a discrete object, or objects, or continuum objects) to somehow influence the probabilities of experiments being performed locally on it. There is also nothing in this specification that prohibits an act of observation or measurement of an object or property from influencing, perturbing and/or even destroying said properties of the object or royally messing up the property.[9] Realism thus assumes that an object's properties determine minimally the probabilities of the results of experiments locally

[9] Whether or not there is a disturbance of a property (or properties) that is made during its measurement certainly does not necessarily mean that there was not a well-defined property of the stuff existing prior to a measurement. Such a disturbance, if present, simply indicates a clumsy measuring apparatus or procedure. At worst, such a clumsy measurement only messes up the properties available for a subsequent measurement. Also, simply because no one at present can think of how to build

performed on it. Realism, under the additional constraint of locality, become Local Realism. Local Realism assumes that the results of said experiments do not depend on other actions performed far away, especially when those actions are performed outside of the light-cone of the local experiment.

Local Realism is the combination of the philosophy of realism with the principle of locality. Local Realism describes "real stuff in real space–time". Stuff, objects, and their associated properties, as referred to here, are what John Bell called Local Beables, and what Einstein Podolsky and Rosen (1935) called "elements of reality". The properties of an object constitute a description of the stuff that is "really there" in nature, independently of our observation of it. When we perform a "measurement" of these properties, we don't really need to know what we are actually doing, or what we are really measuring. The fundamental assumption underlying Local Realism that what is "really there", even if we don't know exactly what it is, nonetheless somehow influences what we observe, even if said influence is inherently stochastic and/or perhaps irreproducible from one measurement to the next. Recall that Einstein et al. (1935) attempted to define an object's properties as something that one can measure, but they further required that the measurement result be predictable with certainty. However, given Ben Franklin's observation that the only predictions that are certain in life are for death and taxes, said definition becomes meaningless, because it describes nothing that can ever be present in reality. Local Realism's definition is much looser and requires no predictions with certainty.

Precisely how did Clauser and Horne (1974) define an object with such extreme generality? For the purposes of Local Realism and its tests via Bell's Theorem and the Clauser–Horne (CH) inequality, a purely operational definition of an object suffices. An object (or collection of objects) is stuff with properties that one can put inside a "box", wherein one can then perform measurements inside said box and get results whose values are presumably influenced by the object's properties.[10] What is a "box"? A box is defined as a closed Gaussian surface,[11] inside of which one can perform said measurements of said properties. Three dimensional (temporally shrinking) Gaussian Surface "boxes" are shown in Fig. 3.8. For Local Realism, such a box becomes a four-dimensional Gaussian surface consisting of the backward light cone (extending to $t = -\infty$) enveloping a three dimensional box, that contains the object(s) being measured, at the time that they are being measured. Four dimensional Gaussian Surface "boxes" are depicted in Fig. 3.9.

Familiar examples of "classical" objects that can be put into boxes are solar systems, airplanes, shoes, trapped clouds of atoms, single trapped atoms, electrons, y-polarized photons, a single bit of information, etc. All of these can be put into a sufficiently large box and have their properties (e.g. color, mass, charge, etc.)

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an improved measuring apparatus that will allow a super-observer to make such a measurement also does not preclude the existence of specific properties characteristic of each point in space.

[10] See especially, Clauser and Horne (1974, Footnotes 10–15).

[11] Gauss showed that a "Gaussian surface" (sometimes called a closed surface) is one that divides all of space into two disjoint volumes, wherein one of these volumes may be called the inside, and the other the outside.

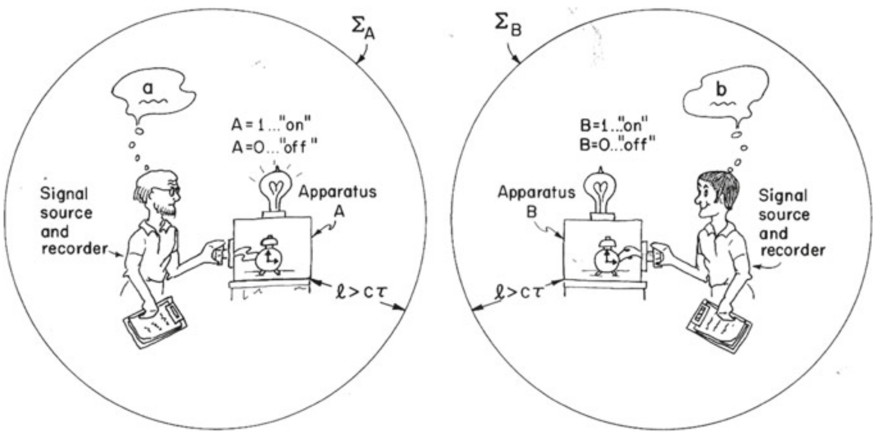

Fig. 3.8 Snapshot of a pair of 3D (temporarily shrinking) Gaussian-surface boxes labeled $\sum_A$ and $\sum_{B.}$ that envelop two separated apparatuses. Each apparatus is measuring a selected property, of an object, giving the binary result, count (on) or no-count (off). The associated property is selected by its associated parameter setting, **a** or **b**. The boxes are space-like separated. Signal sources in each generate the apparatus parameter settings, randomly. In this example, they are generated via human "free-will", with both settings assumed to be independently randomly chosen and not influenced by their communal past

measured. Or can they? Via Bell's Theorem experiments, one may ask—are there examples of objects that cannot be put inside such boxes[12]? If so, such objects cannot be described by Local Realism. Furthermore, if there are parts of nature that cannot be described by Local Realism, then Local Realism must be discarded as a description of all of nature. Sadly, (for Local Realism advocates) experiments now show that the individual particles comprising a quantum-mechanically entangled pair of particles are parts of nature that cannot be described by Local Realism.

Figure 3.8 shows a disjoint, space-like-separated pair of 3D Gaussian-surface "boxes" labeled $\sum_A$ and $\sum_B$.[13] Each box surrounds a pair of objects, each being measured respectively by a pair of measuring apparatuses. Figure 3.9 displays these same two boxes evolving in 2D x-t space. The boxes $\sum_A$ and $\sum_{B.}$ are shrinking at the speed of light as time progresses forward to their positions shown in Fig. 3.8. Two 4D Gaussian-surface boxes, are thus each formed by the backward light cones that envelope 3D boxes labeled $\sum_A$ and $\sum_B$. The 4D boxes contain associated 4D volumes, Γ_A and Γ_B. Figure 3.9 thus shows a (2D projection) space–time diagram of the stuff in space–time, that is contained within these 4D volumes, and that can influence the probability of a count or no-count at each apparatus.

[12] The fact that the simplest possible object—a single bit of information—cannot be put into a "box", in turn gives rise to the field of quantum information.

[13] Figure 3.8 was first presented by the author at the 1976 International"Ettore Majoranna" Conference in Erice, Sicily on "Experimental Quantum Mechanics. The conference was organized by John Bell, Bernard d'Espagnat, and Antonino Zichichi. With present-day jargon, the characters labeled "Signal source and recorder" would now be named Bob and Alice.

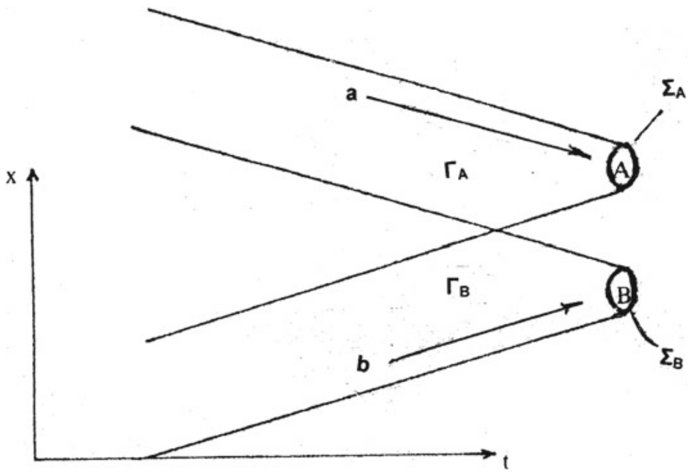

Fig. 3.9 Space–time diagram of stuff in lab space described by Local Realism. Two backward-light-cone 4D Gaussian-surfaces containing associated volumes, Γ_A and Γ_B, respectively, envelope the 3D boxes, $\sum_A$ and $\sum_B$, shown in Fig. 3.8. Objective state properties $\lambda(\mathbf{r}_{lab},t)$ are defined at every space–time point $(\mathbf{r}_{lab},t)$. The probability of a count at apparatus A or B may depend only on properties, $\lambda(\mathbf{r}_{lab},t)$, contained within the associated 4D volume Γ_A or Γ_B. An alternative parameter-selection method is used here with respect to that used in Fig. 3.8. Apparatus parameter settings, **a** and **b**, are now set by signals from events occurring outside of the overlap region of Γ_A and Γ_B. A recent experiment performed by Kaiser and Zeilinger satisfies this condition by having the choices for **a** and **b** set by light from two distant quasars (see Public Broadcasting System, (2018) "*Einstein's Quantum Riddle*", *Nova* television documentary.)

Let us use the symbol, $\lambda(\mathbf{r}_{lab}, t)$, to represent the complete set of properties at any point in space–time, $(\mathbf{r}_{lab}, t)$. It may have whatever level of complexity that is necessary to do so. The properties are assumed to be randomly distributed with an ensemble probability density $\rho(\lambda(\mathbf{r}_{lab}, t))$. Given the arrangement shown in Figs. 3.8 and 3.9 and the assumptions made by Local Realism, the properties $\lambda(\mathbf{r}_{lab}, t)$ located at a point $(\mathbf{r}_{lab}, t)$ that is within Γ_A may influence the probability of a count at apparatus A. Define the incremental probability of a count at apparatus A when the apparatus is configured with parameter setting, **a**, to be $p_A(\mathbf{a}, \lambda(\mathbf{r}_{lab}, t))$. This definition holds, independently of what happens at apparatus B. The corresponding incremental probability of a count at apparatus B with parameter setting, **b**, independently of what happens at detector A, is defined to be $p_B(\mathbf{b}, \lambda(\mathbf{r}_{lab}, t))$, for properties, $\lambda(\mathbf{r}_{lab}, t)$, located at a point, $(\mathbf{r}_{lab}, t)$, within Γ_B. The integrated probability, $p_A(\mathbf{a})$, of a count at apparatus A, irrespective of what happens at apparatus B, that is due the influence of all properties distributed throughout the 4D volume Γ_A is then given by

$$p_A(\mathbf{a}) = \int_{\Gamma A} d\mathbf{r}_{lab}\, dt\, p_A(\mathbf{a}, \lambda(\mathbf{r}_{lab}, t))\rho(\lambda(\mathbf{r}_{lab}, t), \mathbf{r}_{lab}), \tag{3.4}$$

and the probability of a count, $p_B(\mathbf{b})$, at apparatus B, that is due the influence of all properties distributed throughout the 4D volume Γ_B is given by

$$p_B(\mathbf{b}) = \int_{\Gamma B} d\mathbf{r}_{lab}\, dt\, p_B(\mathbf{b}, \lambda(\mathbf{r}_{lab}, t))\, \rho(\lambda(\mathbf{r}_{lab}, t), \mathbf{r}_{lab}). \tag{3.5}$$

Except for correlations caused by common causes in the overlap region, the results at A and B are otherwise independent. The parameter settings **a** and **b** are also chosen independently. Locality then requires that the incremental joint probability of a "coincident" count at both detectors A and B, caused by properties located at a point, $\lambda(\mathbf{r}_{lab}, t)$, is then given by

$$p_{AB}(\mathbf{a}, \mathbf{b}, \lambda(\mathbf{r}_{lab}, t)) = p_A(\mathbf{a}, \lambda(\mathbf{r}_{lab}, t))\, p_B(\mathbf{b}, \lambda(\mathbf{r}_{lab}, t)). \tag{3.6}$$

The factored form, (3.6), is the key ingredient for the CH argument to proceed. The integrated joint probability, $p_{AB}(\mathbf{a}, \mathbf{b})$, of a "coincident" count at detectors A and B for properties located at a space–time point anywhere that might influence the joint result is then

$$\begin{aligned} p_{AB}(\mathbf{a}, \mathbf{b}) &= \int_{\Gamma A \cup \Gamma B} d\mathbf{r}_{lab}\, dt\, p_{AB}(\mathbf{a}, \mathbf{b}, \lambda(\mathbf{r}_{lab}, t))\, \rho(\lambda(\mathbf{r}_{lab}, t), \mathbf{r}_{lab}) \\ &= \int_{\Gamma A \cup \Gamma B} d\mathbf{r}_{lab}\, dt\, p_A(\mathbf{a}, \lambda(\mathbf{r}_{lab}, t))\, p_B(\mathbf{b}, \lambda(\mathbf{r}_{lab}, t))\, \rho(\lambda(\mathbf{r}_{lab}, t), \mathbf{r}_{lab}). \end{aligned} \tag{3.7}$$

Equations (3.5)–(3.7) were used by Clauser and Horne (1974), who found them sufficient to derive the CH inequality as an experimental prediction by all theories in the class defined as Local Realism for the results of an experiment shown with the configuration of Figs. 3.8 and 3.9. The CH inequality is

$$\begin{aligned} -1 \le\ & p_{AB}(\mathbf{a}, \mathbf{b}) - p_{AB}(\mathbf{a}, \mathbf{b}') + p_{AB}(\mathbf{a}', \mathbf{b}) + p_{AB}(\mathbf{a}', \mathbf{b}') \\ & - p_A(\mathbf{a}') - p_B(\mathbf{b}) \le 0. \end{aligned} \tag{3.8}$$

An earlier version of (3.8), the CHSH inequality, was first given by Clauser et al. (1969), for a more restrictive deterministic class of theories. The CH inequality, (3.8) reduces to the CHSH inequality[14] when various modifications are made for the experimental arrangement (the inclusion of source heralding, or Clauser and Horne's no-enhancement assumption) so that the CHSH inequality can also be used for experimental tests of Local Realism.

By carefully defining what "Local Realism" is, Clauser and Horne provide heuristic value in showing what quantum mechanics is not. Local Realism provides experimental predictions that must be obeyed by any theory that attempts to describe "*real stuff in real space–time*". Importantly, the CH prediction (3.8) differs from

[14] Both the CH and CHSH inequalities are examples of what Clauser and Horne (1974) named "Bell Inequalities". See Clauser (2017) for a review of the various testable and tested Bell inequalities.

that made by quantum mechanics. Experimental tests for either the CHSH or the CH inequality are difficult, and it has taken many years for technology to advance to the point where loophole-free direct tests of (3.8) can be performed. In an effort to allow testing with 1970's technology, Clauser and Horne (1974) provided a very plausible auxiliary assumption (their no-enhancement assumption), which allowed the technology of that era to provide an experimental test. Clauser (2017) provides a review of this and other assumptions that have been made to allow experimental testing of Local Realism over the years, as technology for performing such tests has improved.

Starting in 1972 with the first experiment by Freedman and Clauser (1972), followed by the second one by Clauser (1976), and then followed by a long list of confirming experimental refinements continuing to the present, Local Realism has been finally conclusively refuted by experiment. Clauser (2017) provides a partial list of twenty experiments that have been performed at different laboratories around the world during the period 1972 through 2013 to test Local Realism. All but one of these experiments refute Local Realism's predictions. The theory's basic predictions along with "acid tests" of its prohibition of internal super-luminal communication have all been now tested experimentally. Clauser and Shimony's (1978) conclusion regarding Local Realism still stands—"Consequently, it can now be asserted with high confidence that either the thesis of Realism or that of locality (or perhaps even both) must be abandoned. Additionally, any theory that falls within the scope of its underlying assumptions must also be abandoned."

As we shall demonstrate below, the lab-space formulation of quantum must similarly be abandoned.

The Lab-Space Formulation of Quantum Mechanics

Textbooks that promote the lab-space formulation of quantum mechanics generally employ conceptual models that are motivated and justified by the concepts depicted by Figs. 3.1–3.7. Once Schrödinger's equation has been formulated in lab space using this conceptual model, these books then immediately proceed to use it, lab space, and the conceptual model to solve a long list of single-particle problems. The problems typically include free-particle motion, motion of a single particle confined in a 1D and 3D square well, motion of a single particle at a potential step and through a potential barrier, finding the energy levels of a hydrogen atom, quantized orbital angular momentum, the scattering of a single particle by a central potential, and semi-classical radiation theory.

Max Born appears to have been a father of the lab-space formulation of quantum mechanics. It is described in detail in his seminal book, *Moderne Physik*, Born (1933, 1935, 1969). Subsequently authored books by Schiff (1955), Dicke and Wittke (1960), Eisberg (1961, 1967), Merzbacher (1961, 1970), Feynman (1948), Feynman and Hibbs (1965), Eisle (1964), and French and Taylor (1978), all formulate quantum mechanics in lab space, whether they mention this fact or not.

A basic tenant of the lab-space formulation is that Schrödinger's wave function is treated as a "classical field" that is defined and formulated in lab space, as per section "Laboratory Space and Classical Fields". Textbooks that use a lab-space formulation treat Schrödinger's wave function, $\Psi_{Lab}(\mathbf{r}_{Lab}, t)$, as a field that propagates in lab space, as depicted in Figs. 3.1–3.7. Correspondingly, the real-valued probability density, $|\Psi_{lab}(\mathbf{r}_{lab}, t)|^2$, also propagates in lab space as a classical field, like one found in electrodynamics or fluid mechanics, except that it is subject to the somewhat mysterious "statistical interpretation". That "interpretation" is universally attributed to Max Born.

Born, in his textbook (see quote from his book in the section "Born's Argument-Space Ambiguity"), describes the *"wave amplitude",* $\Psi_{lab}(\mathbf{r}_{lab}, t)$*,* as an *"ordinary physical magnitude"*. Born's modification of the traditional classical field concept occurs with his invention of the "*statistical interpretation*". He uses $|\Psi_{lab}(\mathbf{r}_{lab}, t)|^2$ to describe, not only probability density in lab space, but also uses it to describe particle density as a function of lab-space position, wherein particles move at the group velocity of a wave that propagates in lab space. (See the section "Born's Argument-Space Ambiguity" for a detailed discussion of Born's description.) Thus, he uses it to describe the "density" of stuff in lab space, although the term "particle density" is obviously difficult to define for a single-particle theory. His metaphoric usage is perhaps excused by his also calling it a "probability density".

Given the definition of lab space from the section "Laboratory Space and Classical Fields", and the definition of Local Realism from the section "Bell–Clauser–Horne–Shimony Local Realism", then $\mathrm{Re}[\Psi_{lab}(\mathbf{r}_{lab}, t)]$, $\mathrm{Im}[\Psi_{lab}(\mathbf{r}_{lab}, t)]$, and $|\Psi_{lab}(\mathbf{r}_{lab})|^2$ all qualify as classical scalar fields. Unfortunately, the lab-space formalism of quantum mechanics only applies to single-particle systems! It will be shown in the section "Born's Argument-Space Ambiguity" that there is no rigorous method to allow lab-space wave functions, along with a lab-space formulation of quantum mechanics, to be extended to describe $N \geq 2$ particle systems. This impossibility might be expected, since Local Realism's formulation and Bell's Theorem forbids it. Correspondingly, experiments that refute Local Realism also refute a lab space formulation of quantum mechanics.

Single Particle Schrödinger's Equation in Lab Space

Authors formulating Schrödinger's equation in lab space typically start by "searching" for a partial differential equation that is, itself, formulated in lab space. Schiff (1955, pp. 20–22), for example, formulates Schrödinger's equation as a direct analogy to sound waves. Merzbacher (1961, 1970, pp. 34–42), Eisberg (1961, 1967, pp. 166–170, Dicke and Wittke (1960, pp. 23–36), and others rely on conceptual aids like Figs. 3.1–3.7. The lab-space formulation generally starts with a requirement that its solution for a free particle be a plane wave, as per

$$\Psi_{Lab}(\mathbf{r}_{Lab}, t) = \exp\left(i(\mathbf{k} \cdot \mathbf{r}_{Lab} - \omega t)\right), \tag{3.9}$$

with its propagation vector is given by

$$|\mathbf{k}| \equiv 2\pi \,/\, \lambda_{\mathrm{deBroglie}},$$

and where the fundamental relations of quantum theory, deBroglie's and Einsteins's relations,

$$\lambda_{\mathrm{deBroglie}} \equiv \mathrm{h}/|\mathbf{p}|,\ \mathrm{E} = \hbar\omega, \mathbf{p} = \hbar\mathbf{k}. \tag{3.10}$$

also apply. Equation (3.9) then becomes

$$\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}}, \mathrm{t}) = \exp\left(\mathrm{i}(\mathbf{p}\cdot\mathbf{r}_{\mathrm{Lab}}/\hbar - \mathrm{E}\,\mathrm{t}/\hbar)\right). \tag{3.11}$$

The technique employed to obtain the desired partial differential equation is to use a somewhat mysterious but now standard "operator substitutions trick"[15] for these variables, as per

$$\begin{aligned}
&\mathrm{E} = \hbar\omega = \mathrm{p}^2/\,2\mathrm{m} + \mathrm{V} \rightarrow -\mathrm{i}\hbar\,\partial/\partial\mathrm{t},\\
&\mathbf{p} = \hbar\mathbf{k} \rightarrow -\mathrm{i}\hbar\nabla_{\mathrm{Lab},}\\
&|\mathbf{p}|^2 \rightarrow -\hbar^2\nabla^2_{\mathrm{Lab}}
\end{aligned}$$

The equation's component parts are then combined using the indicated operator-substitutions trick to yield the desired partial differential equation—Schrödinger's time dependent equation in lab space,

$$[-\left(\hbar^2/2\mathrm{m}\right)\nabla^2_{\mathrm{Lab}} + \mathrm{V}]\,\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}}, \mathrm{t}) = \mathrm{i}\hbar\,\partial/\partial\mathrm{t}\;\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}}, \mathrm{t}). \tag{3.12}$$

Additionally, the so-called Hamiltonian operator is defined as

$$\boldsymbol{H}_{\mathrm{Lab}} \equiv -\hbar^2\nabla^2_{\mathrm{Lab}} + \mathrm{V}(\mathbf{r}_{\mathrm{Lab}}), \tag{3.13}$$

giving a more compact form of Schrödinger's time dependent equation in lab space,

$$\boldsymbol{H}_{\mathrm{Lab}}\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}}, \mathrm{t}) = \mathrm{i}\hbar\partial/\partial\mathrm{t}\;\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}}, \mathrm{t}). \tag{3.14}$$

Stationary state solutions, $\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}})$, are readily found by factoring out the time dependence using

$$\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}}, \mathrm{t}) = \exp\left(\mathrm{i}(\mathbf{p}\cdot\mathbf{r}_{\mathrm{Lab}}/\hbar - \mathrm{Et}/\hbar)\right) = \Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}})\exp\left(-\mathrm{iEt}/\hbar\right), \tag{3.15}$$

[15] This mysterious trick is referred to by Messiah (1961, p. 885) as the "*Schrödinger correspondence rule*".

whereupon Schrödinger's time-independent equation in lab space is given by

$$\boldsymbol{H}_{\mathrm{Lab}}\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}},\mathrm{t}) = \mathrm{E}\Psi_{\mathrm{Lab}}(\mathbf{r}_{\mathrm{Lab}},\mathrm{t}). \tag{3.16}$$

Equation (3.15) will be recognized as the time-independent Helmholtz equation in lab space, familiar from electromagnetic theory. A solution to (3.15) is then an eigenfunction of the operator $\boldsymbol{H}_{\mathrm{Lab}}$, with the eigenvalue E.

Born's Probability Density and Conserved Probability Current Defined in Lab Space

Born's statistical interpretation starts with his definition of the scalar field,

$$\mathrm{P}(\mathbf{r}_{\mathrm{lab}},\mathrm{t}) \equiv |\Psi_{\mathrm{lab}}(\mathbf{r}_{\mathrm{lab}},\mathrm{t})|^2, \tag{3.17}$$

as the probability density in lab space. A common assertion by Born and echoed by all quantum mechanics books is that is that $\mathrm{P}(\mathbf{r}_{\mathrm{lab}}, \mathrm{t})$ describes a "*wave of probability*". Under the lab space formulation and Born's "statistical interpretation", the probability for detecting the single particle's presence with a detector positioned at $\mathbf{r}_{\mathrm{lab}} = \mathbf{r}_{\mathrm{det}}$, within the differential volume element,

$$\mathrm{d}^3\mathbf{r}_{\mathrm{lab}} \equiv \mathrm{dx}_{\mathrm{Lab}}\mathrm{dy}_{\mathrm{Lab}}\mathrm{dz}_{\mathrm{Lab}},$$

between the times, $\mathrm{t}_{\mathrm{det}}$ and $\mathrm{t}_{\mathrm{det}} + \mathrm{dt}$, is given by

$$\mathrm{P}(\mathbf{r}_{\mathrm{lab}} = \mathbf{r}_{\mathrm{det}},\ \mathrm{t} = \mathrm{t}_{\mathrm{det}}) \equiv |\Psi_{\mathrm{lab}}(\mathbf{r}_{\mathrm{lab}} = \mathbf{r}_{\mathrm{det}},\ \mathrm{t} = \mathrm{t}_{\mathrm{det}})|^2. \tag{3.18}$$

Given Born's statistical interpretation, $\mathrm{P}(\mathbf{r}_{\mathrm{lab}}, \mathrm{t})$ also describes the local particle/probability density in lab-space position. The probability density's normalization is found by integrating the dummy variable, $\mathbf{r}_{\mathrm{lab}}$, over all space, as per

$$1 = \int \mathrm{d}^3\mathbf{r}_{\mathrm{lab}}\ \mathrm{P}(\mathbf{r}_{\mathrm{lab}},\mathrm{t}). \tag{3.19}$$

For a wave function that is not square integrable, like that of a plane wave, a finite normalization volume alternatively may be used. Equation (3.19) specifies that the probability of finding the particle somewhere within the lab-space normalization volume is always unity.

In order to link his probability density to particle density, Born further assumes that the particle moves within the lab with a probability flux that he calls a "probability current". It is described by a classical vector field,[16] $\mathbf{S}(\mathbf{r}_{lab}, t)$. It is defined such that the particle and/or probability flux impinging on a detector's surface is given by $\int \mathbf{S} \cdot d\mathbf{A}$, where d**A** is the detector-surface normal's differential area, and where the integral extends across the detector's surface. Given the normalization (3.19), the appropriate flux-density vector is then just the single-particle's velocity vector[17] defined locally at the point, $\mathbf{r}_{lab}$, as

$$\begin{aligned}\mathbf{S}(\mathbf{r}_{lab}, t) &\equiv \mathrm{Re}[\Psi^*_{lab}(\mathbf{r}_{lab}, t)\,(\mathbf{p}/m)\Psi_{lab}(\mathbf{r}_{lab}, t)\\ &= \mathrm{Re}[\Psi^*_{lab}(\mathbf{r}_{lab}, t)\,(-i\hbar\nabla_{Lab}/m)\Psi_{lab}(\mathbf{r}_{lab}, t)].\end{aligned} \tag{3.20}$$

The second line of (3.20) uses the "standard" operator substitutions trick used above in the section "Single particle Schrödinger's equation in lab space". To show consistency among the definitions, Born relies on the fact that $P(\mathbf{r}_{lab})$, $\mathbf{S}(\mathbf{r}_{lab}, t)$, $\mathrm{Re}[\Psi_{lab}(\mathbf{r}_{lab}, t)]$, and $\mathrm{Im}[\Psi_{lab}(\mathbf{r}_{lab}, t)]$ are all classical scalar and vector fields defined in lab space, as per the definitions given above in the section "Laboratory Space and Classical Fields". Green's theorem then applies to these fields. By using Green's theorem and Schrödinger's equation (3.12) together, Born proceeds to show that the fluid-flow conservation equation,

$$\partial/\partial t\, P(\mathbf{r}_{lab}, t) + \nabla_{Lab} \cdot \mathbf{S}(\mathbf{r}_{lab}, t) = 0\,, \tag{3.21}$$

applies to his probability (and particle) density and flux. Given the conservation equation (3.21), he then refers to the vector field, $\mathbf{S}(\mathbf{r}_{lab}, t)$, as a "conserved probability current". Born (1933, 1935, 1969, Appendix XX) then demonstrates the use of (3.21) by applying it to the problem of flux conservation in Rutherford scattering. It is important to notice that equation (3.21) intimately relies on the use of the lab-space gradient operator, defined above by (3.3).

Born's presentation and definitions became an immediate hit and were universally adopted. They are repeated in many textbooks, regardless of the book's choice of propagation space. Textbooks that give this presentation in lab space include Schiff[18] (1955, pp. 22–24), Dicke and Wittke (1960, pp. 60–62), Merzbacher (1961, 1970, pp. 35–37), Eisberg, (1961, 1967, pp. 172–175) and French and Taylor (1978,

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[16] Schiff (1955) and Dicke and Wittke (1960) use the symbol **S** for the probability current density, while Messiah (1961) uses the symbol **J**.

[17] Dicke and Wittke (1960, pp. 60–62) say, "One may think of this wave as representing a swarm of particles with an average density of one particle per cubic centimeter. In this case, the particles are moving with momentum m**v**, or have a velocity $\mathbf{v} = \mathbf{p}/m$. With this velocity and with an average density of one particle per cubic centimeter, v particles per second pass through a surface area of one square centimeter perpendicular to the direction of motion of the particles. This constitutes the probability flux of the wave."

[18] Schiff (p. 24) says "It is thus reasonable to interpret $\mathbf{S}(\mathbf{r}, t)$ given by Eq. (7.3) as a *probability current density*." Further on, he hedges and says "While this interpretation of **S** is suggestive, it must be realized that **S** is not susceptible to direct measurement in the sense in which P is. Nonetheless, it is sometimes helpful to think of **S** as a flux vector."

pp. 374–378). Textbooks that give the argument in configuration space are discussed in the section "Born's Argument-Space Ambiguity".

The Configuration-Space Formulation of Quantum Mechanics

The configuration-space school is described in John von Neumann's seminal textbook, *Mathematiche Grundlagen der Quantenmechanik*, (1932, 1955), and also in Messiah (1961). Von Neumann's (1932, 1955) textbook, unlike it's contemporary, Born's (1933, 1935, 1969) textbook, promotes the calculation of predictions for the results of experiments via the use of purely abstract mathematical tools with no tangible space–time counterparts. It is formulated using a very general abstract k-dimensional argument space for the wave function that is called configuration space. The quantum mechanical system being described by the school's formalism is very general, and the associated wave function, Ψ_{config}, is similarly very general. It describes a "system" with k-degrees of freedom. The associated Schrödinger's equation can be tailor-made to fit any system by specifying its degrees of freedom and the associated Hamiltonian. The wave function can describe a system comprised of any number, N, of particles, by letting the degrees of freedom be the positions of these N particles. The number of particles in the system may include the special cases $N = 1$ and $N = 0$. By including the $N = 0$ case, the existence of some finite number of particles is thus optional. The configuration-space formalism far surpasses the generality of the systems that can be described by those covered by the lab-space formalism. The latter is limited to describing only single-particle systems. It should be stressed that lab space is never used within the configuration-space formalism, except for specifying the locations of detectors at the end-point of a particle's presumed trajectory.

Configuration Space

The configuration-space school wave function's arguments are provided by configuration space. That space is an abstract mathematical vector space used for specifying the configuration of a general dynamical system. It originated with the Hamilton–Jacobi theory of classical mechanics, and is commonly used for describing the dynamics of a very general system with k degrees of freedom, and especially for describing systems where the forces acting between particles and their constraints are unknown.

Configuration space can be used in either classical mechanics or quantum mechanics. The system may consist of a continuous field (a continuum), such as a fluid's density, where there are no particles present. One way of doing so is to subdivide lab space into very small cells. The necessary degrees of freedom may then consist of the set of field values at each cell. Alternatively, the field's value may be decomposed as a sum of normal modes, and the degrees of freedom then taken to be the modes' expansion coefficients. Configuration space is also useful for describing the dynamics of a (non-continuum) system consisting of N particles. In that case, $k = 3\,N$ degrees of freedom are used to specify the set of N positions of the N particles. For classical mechanical systems, Goldstein (1950, p. 11) in his book *Classical Mechanics*, defines "degree of freedom" as an allowed unconstrained motion along an independent generalized coordinate. In quantum mechanics, configuration space is generalized beyond Goldstein's definition to include motion in energetically forbidden domains, thereby to allow barrier penetration and tunneling. It is also generalized to allow non-classical motions or other variations such as spin, etc. For example, Bjorken and Drell (1964, p. 2) use a configuration-space wave function, $\Psi_{\text{config}}(q_{1,\text{config}}, \ldots, q_{k,\text{config}}, s_{1,\text{config}}, \ldots, s_{k,\text{config}}; t)$, whose arguments are the system's degrees of freedom, and include the variables, $s_{1,\text{config}}, \ldots, s_{k,\text{config}}$, which are the non-classical spin degrees of freedom of the various particles in their system.

Configuration space is adapted for use by quantum mechanics by what is called "transformation theory". Configuration space is a subset of what is called phase space, which further includes the set of k-generalized momenta conjugate to the k-degrees of freedom. An important feature of quantum theory is that Ψ_{config} depends only this subset of variables, e.g. on either the k-degrees of freedom themselves, or on the k-generalized momenta conjugate to them, or on any possible linear transformation among these that provides some other transformed set of k-degrees of freedom.

Messiah (1961, p. 164) succinctly defines the wave function space used by the school by saying "*The wave functions of wave mechanics are the square integrable functions of configuration space, that is to say, the functions* $\Psi_{\text{config}}(q_1, \ldots, q_R)$ *such that the integral* $\int |\Psi_{\text{config}}(q_{1,\text{config}}, \ldots, q_{R,\text{config}})|^2)\, dq_{1,\text{config}}, \ldots, dq_{R,\text{config}}$, *converges.*" … . "*In the language of mathematics, the function space defined above is a Hilbert space. …*".

As a result of its abstract nature, configuration space (and/or Hilbert space) is sufficiently general to include additional degrees of freedom that are not associated with translational degrees of freedom. Such additional degrees of freedom may include the spin degrees of freedom of say the j-th particle. Furthermore, the generality of this abstract space allows Schrödinger's equation to be formulated to describe the dynamics of the spin-degrees of freedom, of say a 2-spin system, without reference to any spatial variables at all. This latter feature is an important asset for calculating the quantum–mechanical predictions for a Bell's theorem CH inequality test.

Despite its great generality, however, configuration space and the associated wave function have important limitations that of are worthy of note:

(1) A position within the lab where the wave function is to be evaluated is not a degree of freedom of the system. Indeed, Ψ_{config}has no specified position within the lab where it is to be evaluated. It has the same value everywhere. Its value depends only on the system's "configuration", i.e. on the system's degrees of freedom.

(2) The various theorems of vector analysis (e.g. Green's theorem(s), Gauss's theorem, Stokes theorem, etc.) do not apply to functions defined in configuration space, since these functions have no lab-space dependence.

(3) There are no discernable waves that propagate in configuration space, like the waves that propagate in lab space. Dicke and Wittke (1960), and Merzbacher (1961,1970) both express alarm about this feature of configuration space (see the section "Born's Ambiguity's Misuse by the Lab-Space Formulation School"). To allow itself to be visualized, an entity like a wave must move in lab space. This fact becomes annoyingly apparent when two particles are needed for describing the helium atom.

(4) Messiah (1961, p. 119) notes that "*The particle associated with the wave generally possesses neither a precise position nor a precise momentum.*" Despite this claim, the single particle configuration-space wave function, $\Psi_{\text{config},1}(\mathbf{r}_{1,\text{config}}; t)$, is defined in terms of the particle's "*precise position" and thus* depends on it's "*precise position.*" This latter difficulty/ambiguity is somehow excused via the use of transformation theory.

(5) Bjorken and Drell (1964, p. 2) state "*The wave function has no direct physical interpretation, however,* $|\Psi_{\text{config}}(q_{1,\text{config}}, \ldots, q_{n,\text{config}}, s_{1,\text{config}}, \ldots, s_{n,\text{config}}; t)|^2 \geq 0$ *is interpreted as the probability of the system having values of* $q_{1,\text{config}}, \ldots, s_{n,\text{config}}$; *at time* t. *Evidently, this probability interpretation requires that the sum of positive contributions to* $|\Psi_{\text{config}}|^2$ *for all values of* $(q_{1,\text{config}}, \ldots, s_{n,\text{config}})$ *at time* t *to be finite for physically acceptable wave functions* Ψ_{config}." Bjorken and Drell, however, never explain the difference between a wave function that "*has no direct physical interpretation*" and a "*physically acceptable wave function*".

The section "Born's Argument-Space Ambiguity" further discusses some of these limitations and their unanticipated effects.

Schrödinger's Equation in Configuration Space

von Neumann (1932, 1955, Chap. 1) formulates Schrödinger's equation in configuration space. His formulation is similar to that by Messiah (1961, p. 71). It is derived using the Hamilton–Jacobi theory framework. Schrödinger's equation for a system with k configuration-space degrees of freedom, $q_{1,\text{config}}, \ldots, q_{k,\text{config}}$ is given first. The configuration space wave function for this system is then $\Psi_{\text{config}}(q_{1,\text{config}}, \ldots, q_{k,\text{config}}; t)$, where $q_{1,\text{config}}, \ldots, q_{k,\text{config}}$ are dummy variables representing the k-degrees of freedom in the function. Schrödinger's equation for this system is

$$\boldsymbol{H}_{\text{config}}\left(q_{1,\text{config}}, \ldots, q_{k,\text{config}}, -i\hbar\partial/\partial q_{1,\text{config}}, \ldots, -i\hbar\,\partial/\partial q_{k,\text{config}}\right)$$
$$\Psi_{\text{config}}\left(q_{1,\text{config}}, \ldots, q_{k,\text{config}}; t\right) = i\hbar\,\partial/\partial t\,\Psi_{\text{config}}\left(q_{1,\text{config}}, \ldots, q_{k,\text{config}}; t\right), \tag{3.22}$$

where $\boldsymbol{H}_{\text{config}}$ is the Hamiltonian operator for the system. In the particular case when the system consists of N mobile particles whose positions are specified by the $k = 3N$ coordinates, $q_{1,\text{config}}, \ldots, q_{3N,\text{config}}$, von Neumann sets $q_{3j\text{-}2,\text{config}}, q_{3j\text{-}1,\text{config}}, q_{3j,\text{config}}$ to be dummy variables representing the Cartesian coordinates of the j-th particle (with $j = 1, \ldots, N$), as per

$$q_{3j-2,\text{config}} \equiv x_{j,\text{config}},\ q_{3j-1,\text{config}} \equiv y_{j,\text{config}},\ \text{and}\\ q_{3j,\text{config}} \equiv z_{j,\text{config}}. \tag{3.23}$$

Using these variables, one may formally define associated configuration space dummy variable vectors each specifying the position of the j-th particle within the lab as

$$\mathbf{r}_{j,\text{config}} \equiv \hat{\mathbf{e}}_x x_{j,\text{config}} + \hat{\mathbf{e}}_y y_{j,\text{config}} + \hat{\mathbf{e}}_z z_{j,\text{config}}, \tag{3.24}$$

as well as with the momentum operator for the j-th particle,

$$\mathbf{p}_{j,\text{config}} \equiv -i\hbar\nabla_{j,\text{config}}. \tag{3.25}$$

Here, the configuration-space gradient operator is defined by

$$\nabla_{j,\text{config}} \equiv \hat{\mathbf{e}}_x x_{j,\text{config}} \partial/\partial x_{j,\text{config}} + \hat{\mathbf{e}}_y \partial/\partial y_{j,\text{config}} + \hat{\mathbf{e}}_z \partial/\partial z_{j,\text{config}}. \tag{3.26}$$

It should be noted that this operator is not at all the same as the lab-space gradient operator, defined above by equation (3.3).

The N-particle wave function for a system of N-particles is denoted by $\Psi_{\text{config,N}}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{N,\text{config}}, t)$. Note that the values of Ψ_{config} and $\Psi_{\text{config,N}}$ depend only upon the values of the various k degrees of freedom, and/or upon the various positions of the N particles in configuration space. Also note that the values of these functions have no explicit dependence on $\mathbf{r}_{\text{Lab}}$. That is to say, Ψ_{config} and $\Psi_{\text{config,N}}$ have the same value everywhere in lab space, and their values (also sometimes called amplitudes) are spatially constant. While the configuration-space gradient operator defined by (3.26), may have a formal symbolic appearance to the lab-space gradient operator defined above by (3.3), it is not the same operator! The configuration-space gradient operator produces the rate of change of its operand $\Psi_{\text{config,N}}$ with respect to a change in the position of particle j, while the lab space gradient operator produces the rate of change of its operand, Ψ_{Lab}, with respect to the lab-space position where said operand is to be evaluated.

It is emphasized that when the system consists of N mobile particles, the wave function, $\Psi_{\text{config,N}}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{N,\text{config}})$, has no evident spatial dependence, and no specified position in lab space where it is to be evaluated. That is, it does not depend on $\mathbf{r}_{\text{lab}}$. Instead, its various arguments indicate its dependence upon the positions, $\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{N,\text{config}}$, of the N particles. It is also important to note that no special status is given to particle 1 (of N) and to the associated first argument of $\Psi_{config,N}$, even for the special case N = 1. Thus, even for a single particle system, the first argument, $\mathbf{r}_{1,config}$, still specifies the position of the one and only one particle, and not the position within in the lab where the function, $\Psi_{config,1}$, is to be evaluated.

As a result, for a system with k degrees of freedom,

$$\nabla_{Lab}\Psi_{config}\left(q_{1,config},\ \ldots,\ q_{k,config};\ t\right) = 0 \tag{3.27}$$

holds, and for a system of N particles

$$\nabla_{Lab}\Psi_{config,N}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config},\ t\right) = 0 \tag{3.28}$$

holds for any N, including for the special case, N = 1.

By using an operator substitution trick similar to that used by the lab space formulation, the Hamiltonian operator for an N-particle (spinless) system is given by

$$\begin{aligned}&\boldsymbol{H}_{config,N}(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config},\ -i\hbar\nabla_{1,config},\ -i\hbar\nabla_{2,config},\ \ldots,\\&-i\hbar\nabla_{N,config})\Psi_{config,N}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config};\ t\right)\\&\equiv\sum_{j=1,N}\left[-\left(\hbar^2/2m\right)\nabla^2_{j,config}+\ V\left(\mathbf{r}_{j,config}\right)\right]+\sum_{j,k=1,N}V_{j,k}\left(\mathbf{r}_{j,config},\mathbf{r}_{k,config}\right).\end{aligned} \tag{3.29}$$

The last term in (3.29) allows for interactions between the particles. Schrödinger's time-dependent equation for an N-particle system takes the form

$$\begin{aligned}&\boldsymbol{H}_{config,N}(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config},\ -i\hbar\nabla_{1,config},\ -i\hbar\nabla_{2,config},\ \ldots,\\&-i\hbar\nabla_{N,config})\Psi_{config,N}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config};\ t\right) =\\&i\hbar\partial/\partial t\ \Psi_{config,N}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config};\ t\right).\end{aligned} \tag{3.30}$$

In the N = 1 special case, the single particle wave Schrödinger's time-dependent equation is

$$\boldsymbol{H}_{config,N=1}(\mathbf{r}_{1,config},\ -i\hbar\nabla_{1,config})\Psi_{config,1}\left(\mathbf{r}_{1,config};\ t\right) = i\hbar\partial/\partial t\ \Psi_{config,1}\left(\mathbf{r}_{1,config};\ t\right). \tag{3.31}$$

Finally, stationary state solutions, $\Psi_{config,N}(\mathbf{r}_{1,config},\ \mathbf{r}_{2,config},\ \ldots,\ \mathbf{r}_{N,config})$, are readily found by factoring out the time dependence, as per

$$\begin{aligned}&\Psi_{config,N}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config},t\right)\\&=\Psi_{config,N}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config},\ \ldots,\mathbf{r}_{N,config}\right)\exp\left(-iEt/\hbar\right)),\end{aligned} \tag{3.32}$$

whereupon Schrödinger's N-particle and single particle time-independent equations in configuration space are given by

$$\begin{aligned}&\boldsymbol{H}_{\text{config,N}}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{\text{N,config}}, -i\hbar\nabla_{1,\text{config}}, -i\hbar\nabla_{2,\text{config}}, \ldots,\\ &-i\hbar\nabla_{\text{N,config}})\,\Psi_{\text{config,N}}\left(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{\text{N,config}}\right)\\ &= \text{E}_{\text{tot}-\text{N}}\Psi_{\text{config,N}}\left(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{\text{N,config}}\right),\end{aligned} \tag{3.33}$$

and

$$\boldsymbol{H}_{\text{config,N=1}}(\mathbf{r}_{1,\text{config}}, -i\hbar\nabla_{1,\text{config}})\Psi_{\text{config,1}}\left(\mathbf{r}_{1,\text{config}}\right) = \text{E}_{\text{tot}-1}\Psi_{\text{config,1}}\left(\mathbf{r}_{1,\text{config}};\ t\right). \tag{3.34}$$

The quantities $\text{E}_{\text{tot-1}}$ and $\text{E}_{\text{tot-N}}$ are energy eigenvalues for single particle and N particle systems, respectfully.

Calculation of Probabilities Using the Configuration-Space Formulation

Messiah (1961, pp. 126–127) shows how to calculate probabilities within the configuration-space formulation using Born's statistical interpretation. The joint probability density for finding the N particles at a specified set of N positions within the lab within in the differential volume elements, $d^3\mathbf{r}_{1,\text{config}} \equiv dx_{j,\text{config}}\, dy_{j,\text{config}}\, dz_{j,\text{config}}$, etc. $d^3\mathbf{r}_{2,\text{config}}$, is found by evaluating the absolute square of the configuration-space wave function at those N positions within the lab. Thus the probability density, $P_{1,2,\ldots,N}$, for finding a particle 1 at a detector that is positioned at $\mathbf{r}_{1,\text{det}}$, and also finding particle 2 at a second detector that is positioned at $\mathbf{r}_{2,\text{det}}$, etc., between times t_f and $t_f + dt$, is found by setting the dummy variable arguments in $\Psi_{\text{config,N}}$, to $\mathbf{r}_{1,\text{config}} = \mathbf{r}_{1,\text{det}}$, $\mathbf{r}_{2,\text{config}} = \mathbf{r}_{2,\text{det}}$, etc. and then by calculating it via

$$\begin{aligned}&P_{1,2,\ldots,N}\left(\mathbf{r}_{1,\text{config}} = \mathbf{r}_{1,\text{det}}, \mathbf{r}_{2,\text{config}} = \mathbf{r}_{2,\text{det}}, \ldots, \mathbf{r}_{\text{N,config}} = \mathbf{r}_{\text{N,det}};\ t = t_f\right)\\ &= \left|\Psi_{\text{config,N}}\left(\mathbf{r}_{1,\text{config}} = \mathbf{r}_{1,\text{det}}, \mathbf{r}_{2,\text{config}} = \mathbf{r}_{2,\text{det}}, \ldots, \mathbf{r}_{\text{N,config}} = \mathbf{r}_{\text{N,det}};\ t = t_f\right)\right|^2.\end{aligned} \tag{3.35}$$

Normalization of this probability density is then

$$\begin{aligned}&\int d^3\mathbf{r}_{1,\text{config}\cdot},\ d^3\mathbf{r}_{2,\text{config}}, \ldots, d^3\mathbf{r}_{\text{N,config}} P_{1,\ldots,N}\left(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{\text{N,config}}, t\right)\\ &= \int d^3\mathbf{r}_{1,\text{config}},\ d^3\mathbf{r}_{2,\text{config}}, \ldots, d^3\mathbf{r}_{\text{N,config}}\left|\Psi_{\text{config,N}}\left(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, \ldots, \mathbf{r}_{\text{N,config}}\right)\right|^2, t)\\ &= 1.\end{aligned} \tag{3.36}$$

For say an N = 2 particle system, the joint probability density for finding the particles at the positions $\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}}$ and $\mathbf{r}_{2,\mathrm{config}} = \mathbf{r}_{2,\mathrm{det}}$ is given by

$$\begin{aligned} &\mathrm{P}_{1,2}\left(\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}}, \mathbf{r}_{2,\mathrm{config}} = \mathbf{r}_{2,\mathrm{det}};\ \mathrm{t} = \mathrm{t_f}\right) \\ &= \left|\Psi_{\mathrm{config},2}\left(\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}}, \mathbf{r}_{2,\mathrm{config}} = \mathbf{r}_{2,\mathrm{det}};\ \mathrm{t} = \mathrm{t_f}\right)\right|^2. \end{aligned} \tag{3.37}$$

The individual probability density for finding particle 1 at position $\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}}$, without caring about the position particle 2 is given by integrating over the possible positions, $\mathbf{r}_{2,\mathrm{config}}$, for particle 2,

$$\begin{aligned} &\mathrm{P}_1\left(\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}};\ \mathrm{t} = \mathrm{t_f}\right) = \int \mathrm{P}_{1,2}\left(\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}}, \mathbf{r}_{2,\mathrm{config}};\ \mathrm{t} = \mathrm{t_f}\right) \mathrm{d}^3\mathbf{r}_{2,\mathrm{config}} \\ &= \int \left|\Psi_{\mathrm{config},\ \mathrm{N}=2}\left(\mathbf{r}_{1,\mathrm{config}} = \mathbf{r}_{1,\mathrm{det}}, \mathbf{r}_{2,\mathrm{config}};\ \mathrm{t} = \mathrm{t_f}\right)\right|^2 \mathrm{d}^3\mathbf{r}_{2,\mathrm{config}}, \end{aligned} \tag{3.38}$$

and the corresponding individual probability density for finding particle 2 at position $\mathbf{r}_{2,\mathrm{config},2}$ is similarly given by

$$\begin{aligned} &\mathrm{P}_2\left(\mathbf{r}_{2,\mathrm{config}} = \mathbf{r}_{2,\mathrm{det}};\ \mathrm{t} = \mathrm{t_f}\right) = \int \mathrm{P}_{1,2}\left(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}} = \mathbf{r}_{2,\mathrm{det}};\ \mathrm{t} = \mathrm{t_f}\right) \mathrm{d}^3\mathbf{r}_{1,\mathrm{config}} \\ &= \int \left|\Psi_{\mathrm{config},\ \mathrm{N}=2}\left(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}} = \mathbf{r}_{2,\mathrm{det}};\ \mathrm{t} = \mathrm{t_f}\right)\right|^2 \mathrm{d}^3\mathbf{r}_{1,\mathrm{config}}. \end{aligned} \tag{3.39}$$

Factorization of Schrödinger's N-particle Configuration-Space Wave Function

An important "factorization criterion" for Schrödinger's N-particle configuration-space wave function is given in many textbooks to argue that the N-particle configuration-space wave function, $\Psi_{\mathrm{config,N}}(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}}, \ldots, \mathbf{r}_{\mathrm{N,config}}, \mathrm{t})$ is a proper generalization of the single particle wave function, formulated in either lab or configuration space. Unfortunately, it really only works for the latter formulation, and even then, it only works in the unrealistic special situation where any and all interaction between these particles exactly vanishes. It is readily demonstrated to apply to both the time-independent and time-dependent configuration-space Schrödinger's equations. The demonstration is usually given for two-particle systems, but it may be generalized to apply to N-particle systems. It is also typically given for systems for which the degrees of freedom are position and momentum, as is done here. However, it also may be generalized to apply to systems using other degrees of freedom, such as particle spin. The argument is given, for example, by Dicke and Wittke (1960, p 111) for single particle wave functions in lab space, and by Messiah (1961, pp. 127–128), for single particle wave functions in configuration space. Messiah's treatment is followed here.

Consider a two-particle system for which the following conditions hold:

(1) The interaction between the two particles vanishes. That is, the interaction potential between them is assumed to be time independent and vanishes for all time, as per

$$V_{1,2}\left(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}}\right) = 0, \tag{3.40}$$

(2) The two-particle configuration-space Hamiltonian has the form of a sum of two terms.

$$\begin{aligned} &\boldsymbol{H}_{\mathrm{config,N=2}}(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}}, -\mathrm{i}\hbar\nabla_{1,\mathrm{config}}, -\mathrm{i}\hbar\nabla_{2,\mathrm{config}}) \\ &\quad = \boldsymbol{H}_{\mathrm{particle1,config,N=1}}(\mathbf{r}_{1,\mathrm{config}}, -\mathrm{i}\hbar\nabla_{1,\mathrm{config}}) \\ &+\boldsymbol{H}_{\mathrm{particle2,\ config,N=1}}(\mathbf{r}_{2,\mathrm{config}}, -\mathrm{i}\hbar\nabla_{2,\mathrm{config}}), \end{aligned} \tag{3.41}$$

where $\boldsymbol{H}_{\mathrm{particle1,config,N=1}}$ and $\boldsymbol{H}_{\mathrm{particle2,\ config,N=1}}$ are the Hamiltonians for the two individual particles.

(3) $\Psi_{\mathrm{config},1}(\mathbf{r}_{1,\mathrm{config}})$ and $\Psi_{\mathrm{config},2}(\mathbf{r}_{2,\mathrm{config}})$ are energy eigenfunctions of the single particle Schrödinger's equations, (3.34) for particles 1 and 2, respectively with respective (single particle) energy eigenvalues E_1, and E_2.

(4) $\Psi_{\mathrm{config,N=2}}(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}})$ is an energy eigenfunction of the N = 2 particle Schrödinger's equation, (3.33) with the energy eigenvalue $E_{\mathrm{tot\text{-}N=2}} = E_1 + E_2$.

When conditions (1)–(4) hold, it is readily shown that the product of the two single-particle eigenfunctions is then an eigenfunction of the N = 2 particle Schrödinger's equation, as per

$$\Psi_{\mathrm{config},1}\left(\mathbf{r}_{1,\mathrm{config}}\right)\Psi_{\mathrm{config},2}\left(\mathbf{r}_{2,\mathrm{config}}\right) = \Psi_{\mathrm{config,N=2}}\left(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}}\right). \tag{3.42}$$

Messiah (1961, p. 128) notes that this "factorization criterion", (3.42), persists in time, so that, if initially at $t = t_0$, the time-dependent two-particle wave function can be factored, then this condition persists for $t > t_0$, thus

$$\begin{aligned} \Psi_{\mathrm{config,N=2}}\left(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}}, t\right) = \Psi_{\mathrm{particle1,config,\ N=1}}\left(\mathbf{r}_{1,\mathrm{config}}, t\right) \\ \Psi_{\mathrm{particle2,config,\ N=1}}\left(\mathbf{r}_{2,\mathrm{config}}, t\right), \end{aligned} \tag{3.43}$$

holds for $t > t_0$. Messiah (1961) also notes that when the wave function thusly factors, then the joint probability density, as defined above by (3.37), also factors,

$$P_{1,2}\left(\mathbf{r}_{1,\mathrm{config}}, \mathbf{r}_{2,\mathrm{config}}\right) = P_1\left(\mathbf{r}_{1,\mathrm{config},2}\right) P_2\left(\mathbf{r}_{2,\mathrm{config},2}\right), \tag{3.44}$$

and the particles are statistically independent. Messiah's "factorization criterion", (3.43), is equivalent to particle independence.

Probabilities of separated measurements specified by (3.43), (3.38), and (3.39) may be compared with similar probabilities calculated via Local Realism using (3.4), (3.5) and (3.7). Note that the particle-independence factorization condition, (3.44), is similar to (3.6), used by Local Realism above in the section "Bell–Clauser–Horne–Shimony Local Realism". There is no similar joint probability of separated measurements for a lab-space formulation because the lab space formulation describes only single-particle systems.

Wave-Function Factorization or Not!

Lab-space formulations of quantum mechanics (e.g. Dicke and Wittke (1960, p 111)) fallaciously use the factorization argument of the preceding section to argue that $\Psi_{config,N=2}(\mathbf{r}_{1,config}, \mathbf{r}_{2,config})$ is a proper generalization of single-particle lab-space wave function, $\Psi_{lab}(\mathbf{r}_{lab})$, so as to allow the lab space formulation to handle N-particle systems. The argument immediately crumbles, however, when one discovers (as Dicke and Wittke note) that by doing so, the wave function's wave-like properties are then lost. Dicke and Wittke, however fail to note that not only is the argument fallacious, it is also incomplete and ignores other even worse problems. A more complete argument shows that, in the fully general case, even when there is no interaction between the particles, then (3.42), (3.43), and (3.44) do not necessarily hold. Similarly, Messiah does not comment that, even if the particles do not interact after $t = t_0$, but if they have ever interacted in the past, even only slightly prior to $t = t_0$, then the N-particle wave function does not factor, and the particles are not statistically independent. An arbitrary choice with t_0 finite, and/or with V_{12} possibly time dependent then belies the possibility that the particles may have been interacting at some time since the beginning of the universe.

More importantly, the factored forms (3.42) and (3.43) are not the only solutions to Schrödinger's equation(s), even when V_{12} is always time independent. Important other solutions exist that apply to entangled particle states, even when the particles are non-interacting. An important property of entangled state solutions then provides the converse of Messiah's independent particle property. That is, entanglement persists in time, even when there is no longer any interaction between the particles. Messiah's argument should correspondingly be modified to read: *Once independent and with no interaction, then always independent. Conversely, once entangled and even with no interaction, then always entangled.*

The usage of the above wave-function factorization argument by authors to bolster their claims that the configuration-space wave function, $\Psi_{config,N}(\mathbf{r}_{1,config}, \mathbf{r}_{2,config}, \ldots, \mathbf{r}_{N,config}, t)$, is an appropriate N-particle generalization of the single-particle lab-space wave function, $\Psi_{Lab}(\mathbf{r}_{Lab}, t)$, thus totally looses its credibility when the existence of these other entangled state solutions is revealed. If Messiah's argument had been completed correctly, then the appropriateness of $\Psi_{config,N}(\mathbf{r}_{1,config}, \mathbf{r}_{2,config}, \ldots,$

$\mathbf{r}_{N,config}$, t) as a generalization of the N = 1 single particle wave function would be immediately found wanting.

Curiously, despite entanglement's absence in the arguments of the above-mentioned authors, it has been around for a long time. For example, Condon and Shortley (1964) formulate Schrödinger's equation exclusively using configuration space. Their application of quantum mechanics is to many-particle systems. They discuss the factorization property, along with entanglement, and provide a practical use for it. Typically, independent single-electron Hamiltonians and factored wave functions are used there to provide an approximate first-guess solution to an N-particle problem. The interaction potential is then used as a small perturbation to provide an improved solution.

Entangled particle states typically occur when there are degenerate solutions to the single-particle Schrödinger's equations, (3.34), as may occur when there are additional degrees of freedom such as spin for the particles. Entangled particles do not need to be so-called "identical particles". The versatility of the configuration space formulation now becomes particularly useful. Suppose that the single particle wave functions, $\Psi_{particle1,config,\ N=1}(\mathbf{r}_{1,config},\ s_1)$ and $\Psi_{particle2,config,N=1}(\mathbf{r}_{2,config},\ s_2)$, include additional degrees of freedom, s_1 and s_2. Suppose also that these degrees of freedom refer to discrete-state variables, such as spin. This degree of freedom for spin ½ has only two allowed states. If allowed integer dummy index values, s_1, s_2. = 1, 2 are used, then both associated indexed values may be displayed together as a two component column vector.[19] Alternatively, the symbols, s_1, s_2, are sometimes used as dummy indicies. They may each take on one of the two values, ↑ or ↓. (Born uses the values + and −.) Finally, assume that the energy eigenstates are degenerate and both particles have the same energy, irrespective of the value of s_j. It is then straightforward to show that

$$\Psi_{config,N=2-particles}\left(\mathbf{r}_{1,config},\mathbf{r}_{2,config}\right) = \Sigma_{s1=\uparrow,\downarrow}\Sigma_{s2=\uparrow,\downarrow}a_{s1,s2}\\ \Psi_{particle1,config,\ N=1}\left(\mathbf{r}_{1,config},\ s_1\right)\Psi_{particle2,config,\ N=1}\left(\mathbf{r}_{2,config},\ s_2\right). \tag{3.45}$$

The form (3.45) is no longer a simple product, as per (3.42), but is instead now a sum of products of eigenfunctions, with associated amplitudes $a_{s1,\ s2}$. In some special cases (and only in these cases), such as when all of the various $a_{s1,\ s2}$ vanish except one, we have a simple product state like (3.42).

The form (3.45) is useful for demonstrating the Einstein–Podolsky–Rosen paradox via Bohm's two entangled-spins *Gedankenexperiment*. (See Clauser (2017).) There, each of the particles may proceed along one of two different paths through an associated Stern-Gerlach apparatus. When, for example, the spatial dependence

[19] This alternative form is used by Born (1933, 1935, 1969, p.188). Referring to spin, he comments "*We can take this new degree of freedom into account formally, by introducing besides the ordinary co-ordinates an additional co-ordinate σ, which can take only two values together; … We thus obtain a wave function which now depends on five co-ordinates: Ψ = Ψ(x,y,z,t,σ). It suggests itself, however, to split up this wave function into two components…*" Nowhere, however, does Born admit that this set of "*five-coordinates*" is, in fact, in configuration space. (See the section "Born's Argument-Space Ambiguity".).

in (3.45) is not needed, and only the spin dependence is needed or relevant, then $\Psi_{\text{particle1,config, N=1}}(s_1)$ and $\Psi_{\text{particle2,config,N=1}}(s_2)$ can be used to represent the single particles' wave functions. The two-particle entangled-state wave function can then be represented as

$$\Psi_{\text{config,N=2-particles}} = \Sigma_{s1=\uparrow,\downarrow}\Sigma_{s2=\uparrow,\downarrow} a_{s1,s2} \\ \Psi_{\text{particle1,config, N=1}}(s_1)\ \Psi_{\text{particle2,config, N=1}}(s_2). \tag{3.46}$$

It should be noted in passing that equations (3.27) and (3.28) hold for both forms (3.45) and (3.46), so that no conserved probability current can be calculated for either of these two particle systems.

It also should also be noted in passing that Messiah does not give any criterion regarding how small $V_{1,2}$ must be for it to be considered "negligible", eventhough his argument requires that it vanish completely, as per condition (1), above. Unfortunately, particle independence will decay exponentially in time, whereby Messiah's persistence in time of particle independence is then actually really only transitory, even for very small $V_{1,2}$. Furthermore, Spitzer (1956) points out that the cross-section for classical Coulomb scattering between two charged particles diverges and becomes infinite at small scattering angles. Thus, all charged particles always continuously interact with each other, by at least a very small amount. Moreover, all charged particles have been thusly interacting with each other since the beginning of the universe. Additionally, a significant fraction of the universe is comprised of charged particles (protons and electrons), so it would seem short-sighted to ignore their interaction completely. Correspondingly, one may conclude, quantum mechanically, that all charged particles are always entangled, at least by a very small amount, especially since $t = t_0$ may be taken arbitrarily to be a time in the very distant past. Given that for any non-vanishing $V_{1,2}$, particle independence decays in time, then entanglement correspondingly grows in time. Eventually, entanglement always wins out over independence. One may correspondingly wonder, "*Is being a little bit entangled like being a little bit pregnant? Perhaps for both cases, the importance of both conditions depends upon how long you wait.*"

Born's Argument-Space Ambiguity

Unfortunately, there are important discrepancies between the lab-space and configuration-space formulations of quantum mechanics. These discrepancies can be traced to originate with a somewhat hidden ambiguity that was introduced by Born (1933, 1935, 1969) in his seminal book. While promoting his "statistical interpretation", and discussing the configuration-space description of two-particle Rutherford scattering, Born (1933, 1935, 1969, pp. 95–96) states

> "*There are grounds for the conviction of the correctness of the principle of associating wave amplitude with number of particles (or probability). In this picture, the particles are regarded*

as independent of each other. If we take their mutual action into account, the pictorial view is to some extent lost again. We have then two possibilities. Either we use waves in spaces of more than three dimensions (with two interacting particles we would have 2 X 3 = 6 coordinates), or we remain in three-dimensional space but give up the simple picture of the wave amplitude as an ordinary physical magnitude and replace it with a purely abstract mathematical concept, (the second quantization of Dirac, Jordan) into which we cannot enter. … This is for us the really important question, for clearly enough the corpuscular and wave ideas cannot be fitted together into a homogeneous theoretical formalism without giving up some fundamental principles of the classical theory. The unifying concept is that of probability."

Born thus takes $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}}, t)$ to be an "*ordinary physical magnitude*" in lab space, i.e. in what he calls "*three-dimensional space*". He also claims that we can "*remain in three-dimensional space*" i.e. use the amplitude $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}}, t)$ in lab space as long as we use his probability "concept". He thereby claims that Ψ_{lab} and $\Psi_{\text{config,N=2}}$, are equivalent, and that both have an "*ordinary physical magnitude*", even though they are differently defined for the two different argument spaces. He asserts that this is true simply because they are both defined to yield the same probability. Added together, he is saying that $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}}, t)$ and $\Psi_{\text{config,N=2}}(\mathbf{r}_{1,\text{config},2}, \mathbf{r}_{2,\text{config},2}, t)$ and their associated argument spaces, $\mathbf{r}_{\text{lab}}$ and $\mathbf{r}_{1,\text{config},2}$, $\mathbf{r}_{2,\text{config},2}$, are equivalent and interchangeable, He doesn't seem to notice that there are, in fact, two (or more) different "*three dimensional spaces*" to choose from—the argument spaces for $\mathbf{r}_{\text{lab}}$ and for $\mathbf{r}_{1,\text{config},1}$—and he correspondingly does not distinguish between them.

Herein he also creates an ambiguity regarding the meanings and uses of the arguments, $\mathbf{r}_{\text{lab}}$ and $\mathbf{r}_{1,\text{config},2}$, $\mathbf{r}_{2,\text{config},2}$, and of the associated wave function's, Ψ_{lab} and $\Psi_{\text{config,N=2}}$. He rejects the idea that either wave function's meaning is that used in quantum field theory (as discussed below in the sections "Quantum Field Theory 1—Quantization of Known Classical Fields" and "Quantum Field Theory 2—Second Quantization of Wave-Functions"), because it is "*a purely abstract mathematical concept*". Motivated by his evident distaste for purely abstract mathematical concepts, and given his proclaimed equivalence of the spaces, Born appears to say that the use of lab space may be chosen simply as a matter of taste. By doing so, he misses the fact that configuration space and the associated wave function, $\Psi_{\text{config,N=2}}$, are also themselves purely abstract mathematical concepts, which he admits that he dislikes.

Born's quote creates an ambiguity. It has thus pronounced the lab-space and configuration space formulations to be equivalent and interchangeable. Taken at face value, any such claim is clearly unfounded. The spaces are indeed very different, as has been demonstrated above. Born and his lab-space-formulation followers similarly appear to have missed (or chosen to ignore) the fact that configuration-space is not as versatile as they may have assumed. It cannot do two incompatible things simultaneously. The wave function's argument clearly does not encompass the ability to represent both the position in lab space where Ψ is to be evaluated, and, at the same time, also simultaneously to represent the configuration of the system being described. Simply put, it cannot be multiply defined. The two formulations and their associated argument spaces are clearly not equivalent. The two wave functions are also not equivalent. In addition, they are not equal. As noted above in the section

"The Configuration-Space Formulation of Quantum Mechanics", there is no spatial dependence for Ψ_{config}. Furthermore, recall that there is no special meaning for the various arguments, especially for the first argument of $\Psi_{config,N=1}$, in the special case N = 1. It is not the same argument as that of Ψ_{lab}. As a result, we have in general, $\mathbf{r}_{lab} \neq \mathbf{r}_{config,1,1}$, even though they are both formally three-dimensional.

Correspondingly, for any number of particles, N, described by $\Psi_{config,N}$, we have (as per Eqs. (3.27) and (3.28))

$$\nabla_{Lab}\Psi_{config,N} = 0, \text{ and } \nabla_{Lab}\Psi_{config,1} = 0. \tag{3.47}$$

Gauss's theorem, Green's theorem and the conservation law (3.21) correspondingly do not apply to $\Psi_{config,N}$, or to $\Psi_{config,1}$. Since the arguments and definitions of the wave functions Ψ_{lab} and $\Psi_{config,N=2}$, and the definitions of their associated arguments, $\mathbf{r}_{lab}$ and $\mathbf{r}_{1,config,2}$, $\mathbf{r}_{2,config,2}$ are all very different, one also must conclude further that

$$\Psi_{lab} \neq \Psi_{config,N=2}, \text{ and } \mathbf{r}_{lab} \neq \mathbf{r}_{j,config,N}, \tag{3.48}$$

hold for any j and/or N, including the special cases j = N = 1.

As we have noted in the section "Introduction - What quantum mechanics is Not", Born's ambiguity is produced by a sneaky slight-of-hand. The ambiguity manifests itself by using the same ambiguous (multiply defined) symbol $\mathbf{r}$ (or symbols $\mathbf{r}_1$, $\mathbf{r}_2$, …) to represent very different quantities (or sets of quantities) with different meanings altogether. Then, via this subterfuge two different equations (or sets of equations) that are formally the same in their appearance(s) are produced and claimed to govern nature. Both equations (or sets of equations) use the common ambiguous symbol (or symbols). Presto, the equations are thereby claimed to be equivalent, when in reality they are not. The misdirection and substitution are done so seamlessly that no one is aware of the prestidigitation that has passed. In particular, the configuration-space single-particle Schrödinger's equations (3.31) and (3.34) and the lab-space Schrödinger's equations (3.14) and (3.16) are not the same, despite their formal similarities. Indeed, they are formulated in very different argument spaces.

Born's Ambiguity's Misuse by the Lab-Space Formulation School

Unfortunately, a lab-space formulation of wave mechanics is found to suffer from several serious deficiencies. Paramount among these is that it can only describe single particle systems. We have just noted that there does not appear to be any rigorous (local) method to allow it to be extended to N ≥ 2 particle systems. As a result, a lab-space formulation cannot describe entanglement, as is needed for describing the spectra of atoms with more than one electron and for Bell inequality tests. The inability of the lab-space formulation (outlined above in the section "The lab-space formulation of quantum mechanics") to find a lab-space description of two-particle systems becomes immediately apparent when one is faced with treating the helium atom.

Lab space advocates (see the section "The Lab-Space Formulation of Quantum Mechanics") formulate Schrödinger's equation, specifically to allow wave propagation in lab space via (3.9). Correspondingly, a lab-space wave function clearly can demonstrate actual wave motion, as is depicted in Figs. 3.1–3.7, for matter waves. Correspondingly, said wave function can be used as a model for experimental observations, as are described, for example, by French and Taylor (1958, 1978). Unfortunately, a configuration-space wave function, like $\Psi_{\text{config,N=2}}$ cannot demonstrate wave motion, as was noted by Dicke and Wittke (1960) and by Merzbacher (1961, 1970). The whole lab-space model depicted in Figs. 3.1–3.7 now begins to crumble.

While attempting to describe the helium atom, Merzbacher (1961, 1970, p. 347) notices the argument-space discrepancy between single particle and two particle wave functions. He comments *"Since Ψ is now a function of two different points in space, it can no longer be pictured as a wave in the naïve sense which we found so useful in the early chapters of this book. Instead, Ψ for two particles must sometimes be considered a wave in a six dimensional configuration space of the coordinates* $\mathbf{r}_{1,\text{config},2}$ *and* $\mathbf{r}_{2,\text{config},2}$*."* Merzbacher, however, does not offer any suggestions for restoring the lost utility of his now discredited early chapters. He also says that viewing a wave function as a description of waves propagating in lab space is naïve. He does not, however, offer a more sophisticated viewpoint.

Similarly, Dicke and Wittke (1960, p. 110) comment *"Hence the wave function has the form* $\Psi = \Psi_{\text{config,N=2}}(\mathbf{r}_{1,\text{config},2}, \mathbf{r}_{2,\text{config},2}, t)$*. Note that this function can hardly be interpreted as a physical wave moving in ordinary three-dimensional space. It has the form of a wave moving in a six-dimensional space. Since this is the analog of the wave function for a one-particle system, it is clear that physical wavelike properties which a single particle wave function exhibits are properties which are to be ascribed to one-particle systems only. In other words, Ψ is a physical wave only to the extent that it can be associated with the motion of single particles."*

Almost all authors of lab-space-formulation based textbooks incorrectly enlist Born's argument-space ambiguity, along with Messiah's particle-independence criterion, as described above in the section "Factorization of Schrödinger's N-particle Configuration-Space Wave Function", to fill the N-particle description void left by the lab-space formulation. Correspondingly, it seems probable that the ambiguity introduced by Born is the cause of the conceptual teachings by the lab-space formulation school and the configuration-space school becoming bifurcated in subsequently authored textbooks. Accordingly, Dicke and Wittke (1960, pp. 110–111) and French and Taylor (1958, 1978, pp. 558–560), proclaim (following Born's quote) that the 2-particle configuration-space wave function, $\Psi_{\text{config},2}(\mathbf{r}_{1,\text{config}}, \mathbf{r}_{2,\text{config}}, t)$, is an appropriate 2-particle "generalization" of the single-particle lab-space wave function $\Psi_{\text{Lab}}(\mathbf{r}_{\text{Lab}}, t)$. They do so by simply swapping dummy index definitions, $\mathbf{r}_{1,\text{config}} \leftrightarrow \mathbf{r}_{1,\text{lab}}$, $\mathbf{r}_{2,\text{config}} \leftrightarrow \mathbf{r}_{2,\text{lab}}$, and ignoring the implications of said swap. Our use herein of "Lab" and "config" subscripts immediately reveals the error in this procedure. The argument spaces for lab-space and configuration-space wave functions are clearly not the same. The meanings of the arguments in the wave functions are also not the same. One wave function clearly cannot be considered to be the same as the other, simply because it has the same number of three- "dimensional" arguments, or because its absolute square yields a probability. The associated lab-space and configuration-space Schrödinger's equations, (3.14) and (3.31), while formally similar in appearance (without the distinguishing subscripts), are actually very different, especially in their meanings. Thus, there is no proper N-particle generalization of the lab-space Schrödinger's equation, (3.14). A lab-space formulation is thus limited to the treatment of single particle motion, as was observed and emphasized by Dicke and Wittke (1960).

Alas, the conceptual model that lab-space formulations promote and use is found to be untenable and unable to describe even a helium atom. One might be tempted to justify keeping the lab-space formulation of quantum mechanics, because it perhaps provides an approximation to a better theory. However, changing from lab space hardly can be considered an approximation of configuration space. Is an apple an approximation of an orange? Nonsense! That justification thus fails miserably. Unfortunately, the price one must pay is (sadly) that the configuration-space formulation requires a highly abstract mathematical formalism that is difficult to understand, and that provides no conceptual model to allow its inner workings to be visualized.

Born's Ambiguity's Misuse by the Configuration-Space School

Another important discrepancy between the two schools that is caused by Born's ambiguity occurs with the derivation and use of his conserved probability current. A requirement for any formulation of quantum mechanics is the conservation of particle number. Born had demonstrated this requirement to hold for Rutherford scattering using the lab space formulation. Configuration space advocates are correspondingly obliged to offer a similar assurance. They generally do so by using Born's ambiguity to demonstrate conservation of particle number, by using what Messiah (1961, pp. 119–122) calls a "*conserved current concept*". Messiah claims "*The property of conservation of the norm has a simple interpretation if one introduces the notion of current*." His derivation follows exactly that by Born, and totally ignores the fact that Green's theorem does not apply in configuration space. He correspondingly ignores Eq. (3.47). Despite this oversight, he and other authors (see, for example, Landau and Lifshitz (1958, 1965)) claim "conservation of probability in configuration space", whatever that means. (Messiah does not tell us what it means.)

Messiah (1961, pp. 222–223) also provides a second, alternative derivation of the conservation equation for the current in configuration space using the classical Hamilton's function S. He notes that "*In the classical approximation,* Ψ_{config} *describes a fluid of non-interacting particles of mass m, (statistical mixture) and subject to the potential,* $\mathrm{V}(\mathbf{r}_{\text{config}})$*, the density and current density of this fluid at each* *point of space are at all times respectively equal to the probability density* P_{config} *and the probability current density* $\mathbf{J}_{config}$ *of the quantum particle at that point.*

On p. 224, he further notes that it is "*… valid for systems with any number of dimensions. The density* $P_{config} = |\Psi_{config}|^2$ *is a well defined function of configuration space; similarly the current* $\mathbf{J}_{config}$ *is a well defined vector field of that space.*"

What is meant by his concept of a "*vector field … in a configuration space … with any number of dimensions*" is never explained. Messiah also defines what he means by a "*vector field in a space with "any number of dimensions*", especially when some of those "dimensions" may correspond to non-classical degrees of freedom like spin. Whatever his definition might be, it is clearly very different from the definition of a vector field in lab-space, as is given above in the above section "Laboratory Space and Classical Fields". Messiah has definitely exploited Born's ambiguity to its limits. It should also be noted that Messiah admits (p. 121), "*Of course, the analogy between this probability fluid should not be pushed too far. All pictures based on this analogy contain no more than the property (IV.11).*" [Messiah's Eq. (IV.11), is the conservation law for a classical fluid flowing in lab space.]

In a fashion similar to that used by Born (i.e. use of the probability current's conservation to explain particle flux conservation in Rutherford scattering), Messiah (1961, pp. 369–380) also uses his current "concept", along with Born's ambiguity to describe particle scattering, and to define scattering cross-sections. His description of particle scattering, along with the associated illustrative Figures [Messiah (1961, pp. 374–375)], however, describes waves propagating in lab space, and particles moving in lab space, similarly to those in Figs. 3.1–3.7, above.

Born's Conserved Probability Current as Re-interpreted Using the Configuration-Space Formulation

As noted above, various textbooks firmly establish their formulation of quantum mechanics in configuration space. These books also discuss probability density and conserved probability current as entities that propagate in configuration space. All follow Born's derivation and (erroneously) ignore the fact that Green's theorem does not apply in configuration space, i.e. they all ignore Eq. (3.47). None of these textbooks ever describes what is meant conceptually by the concept of "wave propagation in configuration space". None of them ever fully define what is meant by the concept of a "*vector field … in a configuration space … with any number of dimensions.*" Also recall the section "Born's Ambiguity's Misuse by the Lab-Space Formulation School" above, wherein it is noted that Dicke and Wittke (1960) and Merzbacher (1961, 1970) both admit that these ideas make no sense at all for $N > 1$ particle systems. It would seem evident that if a wave cannot propagate in configuration space (as Dicke and Wittke and Merzbacher note), it is then difficult to understand how a current can flow in configuration space.

Landau and Lifshitz (1958, 1965, pp. 55–58) also claim (without comment) that a conserved probability current somehow propagates and/or flows in configuration space, without their giving any explanation of what this actually means. Bjorken and Drell (1964, pp. 2–9) derive a conserved probability current in configuration space, and use it to demonstrate flux conservation for the single-particle Dirac's equation. They conclude "*Integrating (1.20) over all space and using Green's theorem, we find* $\partial/\partial t \int d^3x\ \psi^\dagger\psi = 0$, (1.23), *which encourages the tentative interpretation of* $\rho = \psi^*{}_{\text{config}}\psi_{\text{config}}$ *as a positive definite probability density.* " They do not seem to notice the incongruity of their simultaneous claims that while Ψ_{config} "*has no direct physical interpretation, …the probability interpretation requires that the sum of positive contributions to* $|\Psi_{\text{config}}|^2$ *for all values of* $(q_{1,\text{config}}, \ldots, s_{n,\text{config}})$ *at time* t *to be finite for physically acceptable wave functions* Ψ_{config}." How can a wave function have "no direct physical interpretation, but simultaneously be "*physically acceptable*"?

Additional confusion that arises from Born's ambiguity is described in the section "Some Observations Regarding Which School is Proper".

Quantum Field Theory 1—Quantization of Known Classical Fields

The third school of thought used for formulating quantum mechanics is what is called quantum field theory. It may be divided into two forms. The first form, herein called Quantum Field Theory 1, quantizes known real classical fields like light and sound, formulated in lab space for massless particles, that describe known *real stuff in real space–time*. The second form second quantizes abstract fields like wave functions for matter-wave fields, formulated in configuration space for massive particles, and/or hypothetical pseudo-classical abstract fields with no observed counterpart in nature. The second form, herein called Quantum Field Theory 2, is discussed in the section "Quantum Field Theory 2—Second Quantization of Wave-Functions".

There are two immediate applications of Quantum Field Theory 1. The first is to light, electromagnetic radiation, and the electromagnetic field. The second is to sound and vibrational displacements of atoms in solids. A primary purpose of this quantization is to provide these fields with a particle-like character. For light, the particles are photons. For sound, the particles are phonons. The need for this quantization and their associated particles originates with Einstein (1917). He noted that the electromagnetic field needs to be reformulated (quantized), in order to allow its description in terms of particles, that he called "*directional radiation bundles*" (a.k.a. light quanta, a.k.a. corpuscles, a.k.a. photons). Einstein demonstrated that without these *directional radiation bundles,* thermal equilibrium cannot be maintained between the radiation field and a gas of molecules.

The (canonical) procedure for quantizing a classical system, such as a classical field, is well defined, and its application to a lab space-classical field is straightforward. The two well known classical fields, light and sound, are thus readily quantized using it. First, one defines the classical system's "degrees of freedom". For a classical field, space is divided into an infinite number of infinitesimal cells, and the field's values at each cells' lab-space position can be taken to be the fields' degrees of freedom. These values, in turn, can qualify as properties of classical "stuff" at the cell's position. Schiff (1955, p. 344) uses this method. Alternatively, Fermi (1932) expands the field's values using a Fourier series, and the series coefficients are then used for the field's degrees of freedom. Second, one uses the degrees of freedom to form an associated Hamiltonian that defines the total energy of the system. For the classical electromagnetic field, the total classical field energy is well known from electromagnetic theory. For sound waves in a solid, atomic vibrational displacements from rest form a field. Assuming the atoms to be harmonically bound spring-mass systems, the total classical vibrational energy (a.k.a. thermal energy) is also readily calculated. Once the Hamiltonian has been defined, it is used in Schrödinger's equation to calculate the field's quantized dynamics.

The quantization of the electromagnetic field, along with the quantum theory of radiation and quantum electrodynamics, were originally developed by Fermi, Dirac, Heisenberg and Pauli. The essential elements are presented in an excellent early review article by Fermi (1932), who uses the canonical procedure to quantize the electromagnetic field. We shall follow Fermi's (1932) discussion in the section "Quantum Theory of Radiation and Quantum Electrodynamics". Photons are seen to emerge from his formalism. The experimental demonstration that photons act like Einstein's *directional radiation bundles* is outlined in the section "Einstein's Need for Directional Radiation Bundles".

A scrutiny of Fermi's treatment reveals the existence of three very different fields that are sometimes confused with each other via Born's ambiguity. (Fermi does not confuse them!) The first of these fields is the classical field itself. The field's classical value is uniquely defined at every point in lab space. The second field is the quantized version of this field. It is displayed via Fermi's equations (3), (4) and (12) in lab space using quantized mode amplitudes, u_s. The field's mode amplitudes (or values) are used as the degrees of freedom of the field. There are an infinite number of them. A third, related, but very different field is in the form of a function formulated in an abstract vector space. It is the *Schrödinger function* for the system (field plus atom), defined in configuration space by Fermi's equations (35). Its value depends on the field's and atom's degrees of freedom. It is computed as the solution to Schrödinger's equation for the system (Fermi's equations (48)), and it displays the calculated quantum dynamics of the system. Fermi carefully uses the term "*Schrödinger function*" rather than the term "*wave function*" to prevent confusion between this function and the quantized classical field's functional dependence on position.

Fermi's *Schrödinger function's* arguments are in configuration space, to be distinguished from the quantized classical field function's argument, which is the position in lab space where the field may be evaluated. What does the Schrödinger function's *value* mean? Similarly to Schrödinger's so-called wave function in configuration space in standard quantum mechanics, when integrated over all (configuration) space, it gives the particle number density. When only one particle is present, this is the integrated probability density, and it equals one, which, of course, is now equal to the number of particles present. Unlike standard quantum mechanics, in a quantum field theory the number of particles may, however, change with time.

Fermi attempts to show that a causal, *real stuff in real space–time,* behavior for the electromagnetic field is maintained for his quantized field. Unfortunately, Fermi fails to notice that quantization in configuration space brings in the possibility of entanglement, whereupon entanglement of separated photons ruins any hopes for such a result. When entanglement is present, measurements of the field at widely separated positions in lab space unfortunately destroy any residual hopes for a causal, *real stuff in real space–time,* behavior for the quantized electromagnetic field. This fact is quite the opposite of the claims made by Bohr and Rosenfeld as outlined by Heitler (1954, 1957, 1960, pp. 76–86), who consider only the unitary evolution of the electromagnetic field, and do not consider its non-unitary evolution that occurs via von Neumann's collapse process. This important fact was first demonstrated by experimental tests of Local Realism, first performed by Freedman and Clauser (1972). More recent experiments by Gisin (2002) starkly demonstrate the extreme lack of causal unitary evolution for the quantized electromagnetic field that occurs via von Neumann's collapse process. This latter important behavior is discussed below in the section "von Neumann's Collapse of the Entangled Two-Photon Quantized Electromagnetic Field".

Following Fermi's (1932) treatment of the subject, many textbooks have been written and improvements to the theory have been added. They are discussed below in section "Improvements to Fermi's treatment of field quantization". Quantization of sound waves and the emergence of phonons from the formalism is described by Henley and Thirring (1962) and by Kittel (1953, 1956). It is performed similarly to Fermi's procedure for the electromagnetic field, and is not discussed here.

Quantum Theory of Radiation and Quantum Electrodynamics

In Fermi's (1932) description, the quantization of the classical electromagnetic field consists of two parts. The first part quantizes the radiation field. Fermi calls this part of the theory the "*quantum theory of radiation*". In the second part, Fermi attempts to quantize an electromagnetic field of "*the most general type that cannot be constructed by simply superposing plane electromagnetic waves*". Fermi calls this second part of the theory "*quantum electrodynamics*",

Fermi's treatment of the quantum theory of radiation begins by using the values of the electromagnetic vector potential $\mathbf{A}_{lab}(\mathbf{r}_{lab}, t)$ at all points $\mathbf{r}_{lab}$ as the basic degrees of freedom of the system to be quantized. He expands $\mathbf{A}_{lab}$ in terms of a set of Fourier-series standing-wave field modes that are functions of $\mathbf{r}_{lab}$. The modes are assumed to exist in a very large rectangular cavity. At the end of the solution, the walls of the cavity are allowed to expand to infinity.[20] The resulting Fourier transform then transforms the spatial degrees of freedom of the field to allow the Fourier coefficients for these modes to be the new transformed degrees of freedom.

In Fermi's treatment, Maxwell's equations describe the classical electromagnetic field in lab space. The energy density of the electromagnetic field at every point in lab space is defined via Maxwell's equations and the Lorentz force law, consistently with Lagrange's and Hamilton's laws of motion. The total energy integrated over all of lab space is correspondingly used to define a Hamiltonian function in terms of the field's mode amplitudes. A Schrödinger's equation (in matrix form) for the evolution of the mode amplitudes is then formulated. Its solution yield's the *Schrödinger function* that, in turn, displays the system's resulting dynamics. The mode energies are all found to act like the excitations of a simple harmonic oscillator. Quantum mechanically, the simple harmonic oscillator has well-known solutions, consisting of a set of states, whose energy levels are equally spaced by energy intervals, $\hbar\omega$, where ω is the field mode's angular frequency. Each incremental excitation is then *interpreted* to consist of the presence within the quantized electromagnetic field of a particle-like photon[21] with energy $\hbar\omega$. To handle atom–field interactions, Fermi couples the field-mode degrees of freedom to the configuration-space degrees of freedom of one or more atoms. Thus, when the atom's degrees of freedom and Hamiltonian are added to that of the radiation field, he obtains "*the fundamental equation of the radiation theory*".

Fermi attempts to show that his formalism for the quantized electromagnetic field describes *real stuff in real space–time* by showing that it provides reasonable predictions for five important effects. He considers (1) Emission from an excited atom; (2) Propagation of light in a vacuum; (3) A case of interference—the Lippman friges, (4) The Doppler effect, and (5) The Compton effect. Fermi's demonstration of "*Propagation of light in a vacuum*" is particularly interesting. In it, he demonstrates the self-consistency of his formalism with regard to the notion of a wave front propagating at light speed between the two widely separated atoms. Both atoms are coupled to the standing-wave modes of the now infinitely wide radiation field cavity modes. One of the atoms is initially excited, and the other is not. He shows that, following the decay of the first atom, the atom's energy is first transferred to an excitation of a phased-superposition of single photon modes of the radiation field. Fermi then shows that because of this carefully phased-superposition, thereafter the second atom cannot be detected in an excited state until a speed of light travel time between

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[20] A problem with Fermi's method that does not seem to have been addressed by him at the time, is that no mirrored box cavity is present in most labs. Indeed, the interiors of most laboratory apparatuses used to test quantum electrodynamics have light absorbing walls, and not perfectly reflecting walls. However, when experiments are actually performed mirrored box cavity, a surprising new wealth of physics is uncovered. See, Berman (1994), A theory with absorbing walls does not appear to have been formulated.

[21] Concerning the use of the term "*interpret* ", note a big jump in logic that occurs here without experimental justification, especially if the photon, hereby defined, is to have the same particle-like properties required by Einstein's (1917) "directional radiation bundles." (See the section "Einstein's Need for Directional Radiation Bundles".).

the two atoms has elapsed. Fermi thereby shows that the radiation field, which is now comprised of a sum of an appropriately-phased set of infinitely-wide standing waves, can produce the effects of a traveling wave "photon" with light-speed causal effects. This important effect is true, however, only for an N = 1 particle excitation (single photon) of the electromagnetic field. Unfortunately, Fermi did not consider an N ≥ 2 particle excitation of the field, where the hoped for causal effect fails miserably. (See the section below "von Neumann's Collapse of the Entangled Two-Photon Quantized Electromagnetic Field".)

Einstein's Need for Directional Radiation Bundles

Einstein (1917) in his seminal paper "*On the quantum theory of radiation*" introduced the concept of particle-like "photons", that he called "*directional radiation bundles*". In that paper, he considered a gas of molecules interacting with electromagnetic radiation (light), and derived the necessary conditions for thermal equilibrium to be maintained between them, and for the second law of thermodynamics to hold in the face of quantum theory. He pointed out that these *directional radiation bundles* (a.k.a. field quanta, a.k.a. corpuscles, a.k.a. photons) must exist. These *directional radiation bundles* must be emitted and absorbed (a.k.a. created and annihilated) by molecules in a gas totally at random. They must conserve energy and momentum in not two but three important processes—absorption, emission, and stimulated emission. He derived the necessary rate coefficients for these three processes. They are now known as the Einstein A and B coefficients. Einstein thus showed that the classical electromagnetic field must be "*quantized*" in terms of these particle-like bundles.

Einstein's need for these "*directional radiation bundles*" is, in fact, a primary motivation for quantizing the electromagnetic field. Correspondingly, it is a little surprising that a particle like behavior for photons is not discussed at any significant length by Fermi, although he does demonstrate conservation of momentum and energy between the atoms and the field. In this absence, there remained an outstanding need to address the issue of Einstein's required particle-like behavior for the photons. Need for an experimental confirmation of this behavior was further noted by Schrödinger (1927) [see also, Jauch (1971)]. Schrödinger pointed out that that a particle-like photon (a directional radiation bundle) will be either reflected or transmitted at a half-silvered mirror (as required by Einstein, and by the quantum theory of radiation), while a wave-like directional radiation bundle, if governed solely by Maxwell's equations, will be both simultaneously reflected and transmitted at a half-silvered mirror. Schrödinger proposed that an experiment should be performed to find out which of these two predictions is true. A first attempt by Ádám, Jánnosy, and Varga (1955) to test Schrödinger's idea was inconclusive, and resulted in only a null experiment, although this fact was not noticed at the time. A fully conclusive experimental demonstration of Einstein's required behavior was finally performed by Clauser (1974), wherein single photons are indeed observed either to be reflected or to be transmitted at a half-silvered mirror, but not both transmitted and reflected simultaneously.

von Neumann's Collapse of the Entangled Two-Photon Quantized Electromagnetic Field

Fermi's two-atom excitation demonstration of "*Propagation of light in a vacuum*" was intended by him to reveal a reasonable causal behavior of the quantized electromagnetic field. His argument works very "nicely" for single-photon excitations of the quantized electromagnetic field. On the other hand, the definition of the word "nicely" used here depends upon one's point of view regarding whether or not a quantized electromagnetic field retains its ability to describe *real stuff in real space–time*. In Fermi's two atom excitation scheme, a detection of the second atom in its excited state immediately precipitates a von Neumann collapse of the spatially-extended quantized electromagnetic field and of its carefully phased standing wave modes. These modes are infinitely-wide and defined everywhere in space. Nonetheless, they collapse instantaneously everywhere in space to an un-excited zero-photon excited state. Unlike the (perhaps discernable) traveling wave-front in lab space that originally causally induces the second atom's excitation, there is definitely no discernable traveling wave-front in lab space that describes the field's de-excitation via the von Neumann collapse. Unfortunately, the collapse's only observable effect is the field's loss of any ability to perform subsequent other atomic excitation.

So, no harm equals no foul? Yes foul! Fermi's argument definitely does not extend "nicely" for two-photon excitations of the infinitely-wide spatially-extended quantized electromagnetic field. Tests of the CHSH and CH inequalities most often employ two-photon polarization-entangled-state excitations of the field. Consider what happens to such a two-photon excitation under the assumption that the quantized radiation field is assumed to represent CH's *real stuff in real space–time*. When the field polarization of one particle-like photon component of the pair is measured, the other particle-like photon component's polarized field is instantaneously collapsed via von Neumann's collapse process of the quantum state of the field. It is instantly reset everywhere to match the polarization of the measured polarization of the first component photon. This collapse occurs instantaneously without Fermi's light travel time delay having elapsed. Unlike Fermi's causal speed-of-light behavior of the quantized field, von Neumann's collapse process instead occurs instantaneously, with an apparently infinite speed. Following a measurement of the polarization of the first photon's field, immediately thereafter the second photon's field can then be detected in a polarization-parallel state.

Lack of nicety gets worse! Gisin (2002) speculated that von Neumann's collapse process might be a *real causal process in real space–time* within the quantized electromagnetic field, and that a collapse-wave-front correspondingly propagates causally in *real space–time*, with a speed perhaps at or less than that of light. However, he importantly noted that if von Neumann's collapse is assumed to be a *real process*, its behavior is not Lorentz invariant. Thus, the possibly finite speed of any collapse-wave-front depends upon what absolute reference frame the experiment is performed in. Gisin tested his hypothesis and has experimentally set a lower limit to the speed of the collapse-wave-front to be 2/3 10^7 and 3/2 10^4 times the speed of light, depending on whether the choice of reference frame is taken to be that of the local "Swiss Alps", or that of the cosmic background radiation.

Gisin (2002) also noticed that if von Neumann's collapse process is assumed to be a *real process*, it is possible to build an apparatus pair with each apparatus moving with respect to the other and with respect to the two-photon source. Each apparatus then measures the polarization of its member photon from an entangled photon pair. The apparatus motions are carefully designed so that each measurement occurs before the other measurement takes place. Zounds! Neither measurement can then precipitate a von Neumann collapse of the field if collapse is a causal process. Gisin thus notes that "*If each measurement happens before the other, then the quantum correlation should disappear, however large the speed of the spooky action!*" He performed the experiment and observed that the quantum correlation persists! Double zounds. Von Neumann's collapse of the quantized electromagnetic field that occurs because of its being measured really is spooky! But everyone knows that ghosts are not real, don't they? The conclusions that one can draw from Gisin's experiments are that, von Neumann's collapse of the quantized electromagnetic field cannot be viewed as a real causal process! The second conclusion is that, despite Fermi's attempt to demonstrate that a quantization of the electromagnetic field allows it to retain some semblance of a description of CH's *real stuff in real space–time*, is a total flop when field measurements are made at remotely separated points in space. Quantization of the field definitely renders it no longer describable as "*real stuff in real space–time*", and photons, in particular, are also not describable as CH's *real stuff in real space–time*. Of course, that same conclusion was obtained earlier from experiments that test the CH and CHSH inequalities.

Improvements to Fermi's Treatment of Field Quantization

There remain additional outstanding problems with Quantum Field Theory 1. Although some have improved over time. Fermi does show that the quantum theory of radiation is sufficient to explain many of the known experimental results to date. On the other hand, he does not show that field quantization is necessary to explain these experimental results. In the early 1970s, said necessity was called into question by many workers in the field of quantum optics, when many experimental results were found to be readily explained in terms of semi-classical radiation theory. Freedman and Clauser (1972) showed that experiments that measured the polarization correlation of entangled two-photon states did indeed demonstrate the necessity. Clauser's (1974) additional experiments further demonstrate a need for field quantization.

Additionally, Fermi notes that his formalism is not without difficulties. He observes that the second part of the theory, quantum electrodynamics, "…*runs into serious difficulty … every charge has an infinite electrostatic self-energy.*" That difficulty echoes back to his use of perturbation theory while addressing the first part of the theory, the quantum theory of radiation. At the end, Fermi admits "*To all these difficulties no satisfactory answer has been given.*" Fermi's method for addressing quantum electrodynamics was summarized subsequently by Feynman (1962, pp. 3–4). Thirty year later, Feynman commented that Fermi's difficulties (with infinite self-energies) leads to "*one of the central problems of modern quantum electrodynamics.*"

Fermi's treatment of field quantization was chosen for analysis here because it allows a graphic display of the existence of two very different fields,—an abstract vector-space "field", the *Schrödinger function*, whose arguments are in configuration space, that is to be distinguished from the quantized classical field, whose function's arguments indicate the position in lab space where it may be evaluated. This choice is not meant to downplay dramatic advances made in the field of quantum electrodynamics that were made in the late 1940s. The reader is directed to the collection of papers on the subject in the volume Selected Papers on Quantum Electrodynamics, Schwinger (1958) for more details on the subject. Quantization of the electromagnetic field is also described in the latter chapters of some of the quantum mechanics textbooks discussed above, e.g. Schiff (1955, Chaps. XIII–XIV), Merzbacher, (1961,1970, Chaps. 20–22), Messiah (1961, Chap. XXI), as well as in other textbooks that are devoted entirely to the subject, e.g., Bjorken and Drell (1965), Harris (1972) and Heitler (1954,1957,1960).

Some Observations Regarding Which School is "Proper"

Born's argument space ambiguity, once revealed, provides greater insight regarding the differences between Bohr and Einstein in their debate regarding the Einstein Podolsky Rosen (1935) "EPR" paradox. They now appear to have been simply talking past each other, each assuming a different argument space for a two-particle wave function, without recognizing this fact. Bohr insisted that a measurement of one particle of an entangled pair disturbs the whole two-particle (global) wave function, no matter how far apart the particles are. Since there is no space–time description for this global wave function, there is no causality problem associated with the "disturbance" of the kind addressed by Gisin (2002). Bohr was presumably assuming a configuration-space formulated wave function. Configuration-space is a purely abstract mathematical entity—a mathematical (sometimes infinite-dimensional) vector space. As we have noted above, it has no dependence on lab space through which a causal disturbance (e.g. von Neumann's) might propagate. Einstein's thinking was evidently conceptually anchored in lab space. His concept of a wave function was presumably that of $\Psi_{\text{lab}}(\mathbf{r}_{\text{lab}}, t)$, i.e. one formulated in lab space. In such case, any such causal disturbance (e.g. von Neumann's) must obviously propagate as a wave through lab-space. By contrast, in Bohr's case, Ψ_{config} depends only on configuration-space variables, and thus has the same value everywhere in lab space. Its very nature is inherently non-local. Super-luminal wave propagation in configuration-space is not prohibited, since it is totally divorced from lab space. Indeed, as noted above, there is no wave propagation at all, let alone any wave propagation from one apparatus to the other, whereupon non-local wave-function collapse is not inconsistent with special relativity. This fact is true no matter how many particles are being described by Ψ, but it only becomes conspicuous and bothersome when at least two or more widely separated particles are being described by Ψ_{config}, i.e. when its spatially constant value depends on the dynamics of both of these separated particles.

As a result of Born's ambiguity and the two schools of thought that emanate from it, we have thus identified two very different, indeed incompatible, formulations of wave functions that are used by practitioners of the art. Following von Neumann's and Messiah's stern warnings, and following the observation here that lab-space formulations are evidently refuted by experiment, we shall define wave functions that are formulated in lab space as "improperly" formulated, and those formulated in configuration space will be defined here as "properly" formulated.[22] Wave functions are also sometimes called" first-quantized fields, whether or not they were "properly" formulated. Wave functions (in standard quantum mechanics, but not in quantum field theory) conserve particle number, independently of what space they are formulated in. Indeed, Born's conserved probability current (improperly formulated in lab space) is constructed from these first-quantized improperly (lab-space) formulated fields. Indeed, it is the basic purpose of Born's current to ensure particle conservation, as was discussed above in the section "Born's Probability Density and Conserved Probability Current Defined in Lab Space". We have also noted above that Born's ambiguity is sometimes improperly used to interchange the meanings of these fields. When the quantum field theory of the matter-wave field is discussed further in the section "Quantum Field Theory 2—Second Quantization of Wave-Functions", said improper use of Born's ambiguity will be seen to get much worse. Classical fields, as defined in the section "Laboratory Space and Classical Fields" above, are by their very nature, definable "properly" only in lab space.

A revealing of Born's argument space ambiguity also spotlights von Neumann's wave function collapse. For such a collapse to make any semblance of sense, it appears that wave functions must be formulated "properly" in configuration space, as von Neumann and Messiah insist. Said collapse occurs everywhere in space–time when a single particle is detected at a specific point in space–time. Just as with the EPR collapse of a two-particle wave function, a single particle configuration-space wave function must instantaneously collapse when, or even in anticipation of the other particles measurement. Given that the collapse occurs only in an abstract configuration space, there is no need for a super-luminal wave collapse to propagate throughout lab space. *Super-luminal wave propagation in configuration-space is not*

[22] Also, it is noted above that while lab-space formulated wave functions attempt to describe Local Realism's *real stuff in real space–time*, they cannot do so in general, and must therefore be considered *improper*ly formulated.

prohibited in configuration space. Gisin's (2002) experiments demonstrate that fact with stark clarity. The fact that it must collapse in anticipation of a measurement is particularly difficult to understand, only if it is to be viewed as a real process in (lab) space–time.

Abstract configuration space is thus inherently inscrutable by its very nature, as the Copenhagen "interpretation" of quantum mechanics has long professed. It thus allows magic to happen! In a configuration-space formulation of quantum mechanics, no objects are needed to exist as "stuff" in lab space. As seems to happen in stage magic, objects can simply appear and disappear. Since there is no stuff needed to be objectively present in lab space, since there are no objects, then faster than light signaling between non-existent objects, and action at a distance between non-existent objects is not issue. Since the configuration-space wave function does not represent a classical field (as defined above in the section "Laboratory Space and Classical Fields"), its value is not a function of lab space position, and the wave function, Ψ_{config}, itself, has no evident spatial dependence. Correspondingly, a non-causal non-unitary collapse that results from a measurement operation is no longer a conceptual problem. In fact, there are no conceptual problems, because there are no conceptual physical models to deal with. Instead, there are only abstract mathematical concepts to deal with. It seems that one must accept the magic, however distasteful that may be!

Quantum Field Theory 2—Second Quantization of Wave-Functions[23]

In the above section "Quantum Field Theory 1—Quantization of Known Classical Fields", we divided quantum field theory into two forms. The first form quantizes known real classical fields that describe known *real stuff in real space–time*, i.e. stuff with real readily observed classical components—light and sound. The second form second quantizes abstract fields like wave functions for matter-wave fields, and/or hypothetical pseudo-classical abstract fields with no observed counterpart in nature. A basic purpose of field quantization is to unify the description of (first) quantized classical fields and second-quantized wave functions, in such a way that allows a varying number of particles to be somehow associated with these various and diverse fields, and to do so in a manner that allows particle creation and annihilation.

[23] Definitions of the terms "*first and second quantization*" as used here are those given by Schiff (1955, p. 348) in his section 46 "*Quantization of the nonrelativistic Schrödinger equation*". He says "*This application implies that we are treating Eq. (6.16)as though it were a classical equation that describes the motion of some kind of material fluid. As we shall see, the resulting quantized field theory is equivalent to a many particle Schrodinger's equation, like (16.1) or (32.1). For this reason, field quantization is often called second quantization; this term implies that the transition from classical particle mechanics to Eq. (6.16)constitutes the first quantization.*" Bjorken and Drell (1965, Chap. 13) similarly use the term in "Second Quantization *of the Dirac equation*", while they discuss in their Chap. 14. "*Quantization of the electromagnetic field*".

Unfortunately, it appears that said unification can occur only with a generous use of Born's ambiguity, given the wide variety of fields that various authors try to sweep together.

In the section "Quantum Theory of Radiation and Quantum Electrodynamics" we described how Fermi quantized a known classical field, the electromagnetic field, in order to produce photons. Doing so, he prescribed the need for two very different field types—those whose arguments are defined in configuration space, and those whose arguments are defined in lab space. The first is the abstract vector-space function that he named the *Schrödinger function*. Its arguments are in configuration space. It is clearly not a classical field, as defined above in the section "Laboratory Space and Classical Fields". It is to be distinguished from the quantized and non-quantized classical electromagnetic fields, whose function's arguments indicate the position in lab space where they may be evaluated. These very different field types are sometimes confused with each other via Born's ambiguity. Indeed, they have a direct parallel to the two schools of thought we have identified and distinguished in the sections "The lab-space formulation of quantum mechanics" and "The Configuration-Space Formulation of Quantum Mechanics". The primary distinction between them is in regards to their associated argument spaces.

There at least 7 mathematically distinct kinds of fields noticed in the reviewed literature on quantum field theory. Some are considered worthy of consideration for field quantization by various authors. Some are actual classical fields that describe known *real stuff in real space–time*. Some are abstract functions in a vector space. Some are totally hypothetical in nature. Following the above definition of "proper", it is noteworthy that some are "properly" defined, and others are "improperly" defined. They include the following fields:

1. Classical fields <u>defined properly only in lab space</u>. These include the classical electromagnetic field that gives rise classically to light waves and quantum mechanically to photons, and the field of vibrational displacements that gives rise classically to sound waves and quantum mechanically to phonons. When acting classically, these fields can host real wave motion in lab space.
2. Quantized version of field #1.(quantized as per Fermi (1932). It is <u>defined properly only in lab space</u>. It can undergo non-causal non-unitary von Neumann collapse. Case 1 fields cannot similarly collapse.
3. Fermi's "*Schrödinger function*" for case #2 quantized atom(s) plus field. It is a vector-space function <u>defined properly only in the configuration space</u> of the atom or atoms and the field degrees of freedom (as per section "Quantum Theory of Radiation and Quantum Electrodynamics"). A *Schrödinger function* is similarly definable for the atom or atoms alone and for the field alone when these entities do not interact.
4. Single-particle wave function (a.k.a. the matter-wave field) <u>defined improperly in lab space</u> for a single particle, that is a solution to the <u>lab-space improperly defined</u> Schrödinger, Klein Gordon and/or Dirac equations. It is the field depicted in Figs. 3.1–3.7. It cannot be <u>properly</u> generalized to describe N particle

systems (as per the section "Quantum Theory of Radiation and Quantum Electrodynamics"), although Schiff (1955, p. 341) claims that field quantization provides a proper method for doing so.[24] Henley and Thirring (1962), p. 3) also consider such fields as suitable for field quantization.

5. Pseudo-classical,[25] "classical free field", similar to a case 4 field, that satisfies, say, the Klein–Gordon equation. Such fields have no observed counterpart in nature. The field is defined in lab space, although it is totally ambiguous as to whether this definition is proper or improper. It is used by various authors, e.g. by Bjorken and Drell (1965), and Messiah (1961, p. 960), to demonstrate methodology for second quantizing fields with no known classical counterpart, whereupon there is no classical way to calculate the associated energy density. Bjorken and Drell (1965, p. 34) note that "*Classically, the field φ(x) is observable and its strength at a point x can be measured.*" Many authors further consider functions and define pseudo-classical fields that are typically permanently complex and/or have non-geometrical abstract properties, such as Dirac spinors. Sometimes, authors quantize a "real scalar field" and seem to play ambiguously on words. They use the word "real" to mean real-valued, i.e. having no imaginary component, rather than known to really exist in nature. (Are ghosts real or imaginary, or even complex?) Various authors thus stretch the limits of one's imagination regarding what may be considered "classical", and/or "real".
6. Single-particle wave function (a.k.a. the matter-wave field) defined only properly in configuration space. It is related to the N particle wave function using a sum of products form, as per sections "Factorization of Schrödinger's N-particle Configuration-Space Wave Function" and "Wave-Function Factorization or Not!". In actuality, it is already a case 3 "Schrödinger function", as per Fermi's definition and usage.
7. N-particle wave function (a.k.a. the matter-wave field), defined only properly in configuration space. In actuality, it is a "Schrödinger function" as per Fermi's definition and usage. Recall that Bjorken and Drell (1964, p. 2) state that properly defined vector-space wave functions that are used as N-particle wave functions have *no direct physical interpretation*. Nonetheless, Bjorken and Drell (1965, Chap. 13) second quantize them (but only when the particles are Fermions).

Case 1 quantization of both light waves and sound waves, properly defined in lab space, discussed above in the section "Quantum Field Theory 1—Quantization of Known Classical Fields", seems to make sense. First quantization of particle motion, properly defined in configuration space, as discussed above in the section

[24] Schiff (1955, p. 341) says "*The field quantization technique can also be applied to a Ψ field, such as that described by the non-relativistic Schrödinger Eq. (6.16)or by one of the relativistic Eqs. (42.4)or (43.3). As we shall see (Sec.46), this converts a one-particle theory into a many particle theory; … Because of this equivalence, it might seem that the quantization of the fields merely provides another formal approach to the many-particle problem. However, the new formalism can also deal with processes that involve the creation and destruction of material particles.*"

[25] Let's pretend that such a classical field exists although we have no evidence that it does.

"The Configuration-Space Formulation of Quantum Mechanics", also perhaps makes sense. It is totally ambiguous as to whether Case 5 first quantization of pseudo-classical fields makes sense, given their vague definition. But, as we have argued in the section "Quantum Field Theory 1—Quantization of Known Classical Fields", a configuration-space wave function is not a classical field in lab space, otherwise it cannot undergo von Neumann collapse. Thus, it seems totally improper to second quantize case 6 and case 7 abstract configuration-space wave functions. Indeed, the only means to do so appears to be to use Born's ambiguity to convert these to be case 4 wave functions or case 5 pseudo-classical wave functions, improperly defined in lab space, and then to proceed by pretending them to be classical fields. Born's ambiguity correspondingly appears to be an integral foundation of quantum field theory of the matter-wave field. Unfortunately, it now appears also to be a major impediment.

If the goals of classical-field quantization and second quantization of wave-functions are to describe real-stuff in real space time, as Fermi apparently has tried to do with the classical electromagnetic field, in terms of a variable number of spontaneously forming particles, then that goal evidently cannot be achieved for any kind of field. It seems that the best that one can hope for is to describe totally abstract stuff (including pseudo classical fields) in real space time with a variable number of spontaneously forming particles. The new remaining problem identified here is how is one to resolve Born's ambiguity *vis à vis* second quantization of mater-wave wave functions.

Conclusions

What then does quantum mechanics not describe? Despite pretenses made in various quantum mechanics books, quantum mechanics does not describe the dynamics of any thing that is objectively real. Have we gotten any closer to answering the question posed in the opening paragraph of this paper?—What do standard quantum mechanics and quantum field theory describe? It seems that the answer is—Not really. What quantum mechanics and quantum field theory seem to describe is something other than objectively real stuff evolving in space–time. It seems that the best that an *improperly* (and inconsistently) formulated form of quantum field theory for matter-waves can offer is a description of totally abstract stuff (including pseudo-classical fields) that evolves non-causally in real space time with perhaps a variable number of spontaneously forming particles.

The basic conclusion obtained from a consideration of the experimental refutation of Bell–Clauser–Horne–Shimony Local Realism is that any theory that describes *real stuff* evolving causally in *real space–time,* must violate experiment for its predictions for separated entangled systems, assuming that it is capable of making predictions for these systems. Lab-space formulations of standard quantum mechanics are unable to make the necessary predictions. Lab space formulations of standard quantum mechanics qualify as theories of Local Realism; however, such formulations of quantum mechanics are unable to make the necessary predictions, thereby to reveal their own "dirty laundry". In any case, lab-space formulated standard quantum mechanics provides, at best, an "*improper*" description for matter waves, where "propriety" instead requires the formulation in configuration space only.

A dilemma is thus revealed by this article—how does one find an "*unambiguous*" and "*proper*" method within standard quantum mechanics and quantum field theory to provide a variable particle number for massive particles that have no known associated classical field. To be "*unambiguous*", the method must avoid use of Born's ambiguity, which, in turn, erroneously considers configuration space and lab space to be equivalent and interchangeable. Unfortunately, at present, Born's ambiguity appears to be an integral foundation of quantum field theory of the matter-wave field for massive particles.

Related questions are—can we live with Born's ambiguity, and/or equivalently, can we experimentally distinguish between the different lab-space and configuration-space formulations of quantum mechanics and quantum field theory? That is, can we have it both ways? It seems that we cannot. The configuration space formulation seems to be required, as per the experimental refutation of Bell–Clauser–Horne–Shimony Local Realism. The answer leaves open the remaining question identified here—How does one resolve Born's ambiguity *vis à vis* second quantization of matter-wave wave functions.

Sadly, for those of us who had hoped to find a theory of *real stuff in real space–time*, we are left with the situation wherein quantum mechanics describes abstract, impossible-to-be-real stuff. Said stuff is described by an abstract mathematical framework, wherein said stuff somehow moves and propagates perhaps stochastically (if the words "moves" and "propagates" still retain any understandable meaning) in an abstract multi-dimensional mathematical space that may contain non-classical abstract components. Whatever quantum mechanics does indeed describe is sadly very difficult to visualize.

There is one final perhaps important irony regarding the "*demise*" of any theory that attempts to describe *real stuff in real space–time*. Said *demise* now includes lab-space formulations of quantum mechanics and quantum field theory. Both Albert Einstein and Max Born[26] were a strong proponents of the quest for a lab-space description (i.e. a. space–time description) of natural phenomena. Correspondingly, said *demise* must certainly have an impact on one's conceptual understanding of Einstein's theory of general relativity, if and when it is reconciled with quantum mechanics. General relativity certainly seems to be a theory within the bounds of Local Realism. Indeed, a fundamental tenant of general relativity is that the geometry of space–time depends on the mass energy content of how much *real stuff* is present

[26] This article has concentrated on Born's quest for a space–time description of natural phenomena. It is noteworthy that the first printing of Born's (1933) seminal textbook predates the famous "EPR" paper by Einstein, Podolsky, and Rosen (1935).

in *real space–time*. Similarly, Stephen Hawking's claims that information is always conserved and is contained in an accessible volume of space–time, especially at a black-hole boundary, seems to be based a use of Born's conserved probability current, which we have noted above is not really conserved in the real space–time framework of general relativity.

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